F. C. BOON

COMPANION TO SCHOOL MATHEMATICS

With a foreword by
A. P. ROLLETT

LONGMANS

COMPANION TO SCHOOL MATHEMATICS

This book is now being included in our list once again with a foreword by A. P. Rollett and a new Bibliography. There have been only minor revisions to the text, for Boon's book has a timeless quality which makes extensive re-editing unnecessary.

This is not a textbook but a book which explores a pleasurable side of mathematics. It is a mine of information for anyone engaged in the study or teaching of mathematics, and was written by a teacher who knew from experience that digressions into history, etymology, paradoxes and glimpses of the way ahead pay handsome dividends. One of the secrets of his success as a teacher was that the interest of his pupils, even the mathematical duffers, was aroused by the expectation of interesting sidelights on the subject. This is a book *par excellence* for any teacher wishing to enrich the content of his lessons, while pupils can dip into it with the certainty of finding something interesting. It should naturally find a place on the shelves of school, training college or University libraries, and has indeed been one of the books recommended by the I.A.A.M. for library use. The book is such a mine of information and so wide in its scope, that it may be read and used with profit and enjoyment wherever mathematics is regarded as an essential part of education, whether here or overseas.

A COMPANION TO SCHOOL MATHEMATICS

By the same author

PUZZLE PAPERS IN ARITHMETIC (*G. Bell & Sons Ltd.*)

A COMPANION TO SCHOOL MATHEMATICS

F. C. BOON, B.A.
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1724

LONGMANS

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FOREWORD

Frederick Charles Boon was born in 1875. He was educated at Nottingham High School, and at Clare College, Cambridge, where he was a scholar and prizeman. After teaching at King's School, Worcester, King Charles I School, Kidderminster, and a Training College (Trinity College, Carmarthen) he joined the staff of Dulwich College in 1903. In 1909 he became Principal Master in Mathematics. He died in Dorset in 1939, three years after his retirement, to the great grief of all who knew him. He was a familiar figure at meetings of the Mathematical Association, endearing himself to all by his kindliness, his whimsical humour, and his defence of the mathematical duffer, to whom he devoted a large part of his teaching time and from whom he professed to have learnt much.

Much of his philosophy of teaching is sketched in his own preface to this book. He knew from experience—and he preached only what he practised—that digressions into history, etymology, paradoxes, and glimpses of the way ahead pay handsome dividends. He used to say that one of his colleagues who ' wasted ' more time in this way than any of the others also got more work done on the syllabus. The secret, he felt, was that boys went into his room with an expectation of something interesting and pleasurable; their attention being more readily enlisted the time was more productive.

This book was one of a remarkable series published by Messrs. Longmans between 1910 and 1925, a period which included one Great War and its depressive aftermath. The series included Nunn's ' Algebra,' Carslaw's ' Non-Euclidean Geometry,' F. S. Carey's ' Calculus,' Miss Hudson's ' Ruler and Compasses ' and Miss Punnett's ' Groundwork of Arithmetic '—books whose influence was and is considerable though all are now out of print.

It is good to see the present book available again, at a time of resurgence of interest in Mathematics and of an insatiable demand for mathematicians of all kinds. This is the book *par excellence* for the teacher striving to find a means of enriching the content of his lessons. It could well be dubbed the mathematics teacher's Boon Companion! Student teachers will find the book a mine of information and suggestion, senior pupils can dip into it with the certainty of finding something of interest, and school mathematics clubs will find it a source of topics for discussion. This book has for many years been on every list of recommended titles for teacher's libraries and school libraries on both sides of the Atlantic; at last the frustating letters ' O.P.' can be, at least temporarily, erased.

A. P. ROLLETT

PREFACE

THIS book is intended to be a companion to the elementary mathematics taught in schools.

It demands little more on the part of the reader than a knowledge of the elementary mathematics of a university matriculation syllabus and of some trigonometry; and for the greater part of the book much less than this.

Not infrequently students who have acquired this knowledge, especially if they have done so under the pressure of examination requirements, have kept too rigidly to the straight route mapped out by their syllabus. The desire to explore byways with alluring vistas has been fearfully repressed lest the goal should not be reached in time; and those ideas which are part of the instinct and training of the specialist have been excluded from their outlook.

An examination certificate is a poor compensation for a starved appreciation of the subject, and even if the certificate is a desirable or necessary objective, it may be the more easily attained if time is spared for wider exploration. The time is not wasted; in the end, experience shows, time is gained. Meanwhile a wider country opens to view; from some eminence, remote from the high-road, the explorer obtains a better observation of his journey and its destination, and glimpses open to him of spacious prospects which are not for those who march in the dust between high hedges. The author's aim and hope have been to provide a sort of a guide book whereby the student may be led to some such points of vantage.

From the time when the pupil begins arithmetic, questions connected with the history of number and of methods of counting and computation, with the meaning of terms and the origin of symbols, invite consideration. As he proceeds, ideas of symmetry, degree, continuity, and so on present themselves and ask to be invested with precision and amplification. At each revision there occur opportunities of co-ordinating ideas and experience, and at every stage the names of mathematicians occur, and the history of mathematical topics enlists his interest. Need it be said that to succumb to the temptation to follow these side-tracks is not to waste time, but to find in the subject a livelier interest—an interest that is cumulative and productive?

In this book are collected and amplified notes on the textbook work as it proceeds through the school course; talks on various ideas that are woven into the fabric of all mathematics; ideas in

which the orderliness and unity of the subject and much of its essential thought and character are centred. The book is divided into chapters allotted to different topics ; but it is clearly impossible to separate by sharp divisions ideas which not only bind together the whole subject, but are also interwoven with one another. There is overlapping and merging, just as in the landscape of our fancy side-paths intersect or lead to a common spot.

The author hopes that the teacher may find the book useful as a Reference Book for his desk ; that the specialist-pupil may find in it an introduction to more advanced work, and more especially that the non-specialist may by its means be helped to realize something of the interest and attraction that the subject has for the specialist.

The author wishes to record his indebtedness to a number of the works mentioned in the Bibliography, especially those of W. W. Rouse Ball ; to the general editors for their criticism and advice when the book was taking shape, and especially to the late Mr. C. S. Jackson, who was then one of them, for the help and encouragement that his friendship freely offered and that his knowledge and experience rendered of inestimable value ; and finally to Mr. P. C. Unwin and Mr. G. I. Sinclair for help and criticism in the proof-reading.

F. C. B.

CONTENTS

CHAPTER I

BIOGRAPHICAL NOTES

There is properly no History, only Biography.—EMERSON.

THE following list includes those mathematicians whose names occur in the elementary parts of the subject. Only brief mention can be made of them. Fuller information is given in Rouse Ball's *History of Mathematics*, and good accounts of a few great men are given in Sir Oliver Lodge's *Pioneers of Science.*

This list should be used by the reader in connexion with Chapter II, which it supplements.

Abel (Niels Henrick Abel; *b.* Findoe, in Norway, August, 1802; *d.* Arendal, April, 1829). Proved that algebraical equations of higher order than the fourth could not be solved in terms of purely algebraical symbols.

Adelhard, of Bath, disguised as a Mohammedan student studied at the Moorish University of Cordova *c.* A.D. 1120, and, obtaining an Arabic copy of Euclid, was the means of introducing its study into mediaeval Europe.

Ahmes (*The moon is born*; the Gk. form is *Amasis*), an Egyptian, wrote the earliest known work on Mathematics (*see* p. 19).

Alcuin (*b.* York, *c.* 735; *d.* Tours, 804). A teacher in the Cloister School in York; assisted Charlemagne in establishing schools on the Continent; reputed author of an arithmetic—*Problems for Quickening the Mind.*

Alkhwarizmi (Mohammed ibn Musa abu Djefar of Khorassan). Wrote an Algebra, *c.* 830, based on that of Brahmagupta. The title ALGEBR W'ALMUKABALA, meaning *restoration and simplification*, contains the origin of the word Algebra (*see* p. 22). Algorism, the mediaeval name for Algebra, is a corruption of his name, which means "Man of Khorassan."

Apollonius (of Perga, in Pamphylia, 260–200 B.C.). Wrote a complete treatise of the Conic Sections. He studied and probably taught at Alexandria.

Archimedes (Syracuse; *b.* 287; *d.* 212 B.C.). One of the greatest of Greek mathematicians. Studied at Alexandria; made the first considerable contribution to the Science of Mechanics and Hydrostatics; the first to undertake a systematic evaluation of π; found the area of a segment of a parabola and of the surface of a sphere. The part he played in the defence of Syracuse against the Romans and the manner of his death, as related by Livy, are well known. In 75 B.C. Cicero visited his neglected tomb and found on it the figure for the demonstration of the area of the surface of a sphere.

Aristotle (*b.* Stagira, 384; *d.* in Eubœa, 322 B.C.). The greatest of Greek philosophers, his authority shackled the mediaeval schoolmen. He included some mathematics in his works, represented unknown quantities by letters, and suggested the Theory of Combinations.

Atwood (George Atwood; 1746–1807). A Cambridge mathematician, in whose TREATISE ON THE RECTILINEAR MOTION AND ROTATION OF BODIES is given a description of Atwood's machine.

Bacon (Roger Bacon; *b.* Ilchester, 1214; *d.* Oxford, 1294). Known in the Middle Ages as a magician, was a pioneer in natural science, and far in advance of his day in mathematics and astronomy.

Barrow (Isaac Barrow; *b.* London, 1630; *d.* Cambridge, 1677). The first Lucasian Professor at Cambridge; the first to use the procedure of the Differential Calculus in the geometry of curves. As a boy at Charterhouse he was so troublesome that his father was heard to pray that if it pleased God to take any of his children he could best spare Isaac. Newton was his pupil, and to Newton, whose genius he generously recognized, he resigned his Chair.

Bolyai (John Bolyai; 1802–1860). A Hungarian officer who at the age of twenty-one had established the hyperbolic geometry (p. 30).

Bramagupta (*b.* 598). A Hindu mathematician of note.

Briggs (Henry Briggs; *b.* near Halifax, 1561; *d.* Oxford, 1631). The first Gresham Professor (London) and Savilian Professor (Oxford); largely responsible for the introduction of logarithms, and the first to compile logarithms to the base 10.

Cardan (Girolamo Cardan; *b.* Pavia, 1501; *d.* Rome, 1576). A charlatan and a scoundrel, but a great algebraist; discussed negative and complex roots of equations; his solution of the cubic, which he published in his *Ars Magna*, 1545, he had obtained from Tartaglia under a solemn oath not to divulge it. A celebrated maker of horoscopes, he became astrologer to the Papal Court, in which post, having made a prophecy that he would die on a certain day, he committed suicide (so a story goes) to ensure its fulfilment.

Ceulen (Ludolph van Ceulen; Cologne, 1539–1610). Calcu-

lated π to thirty-five places. After him German writers called π the LUDOLPHIAN NUMBER.

Ceva (Giovanna Ceva; Milan, 1648–1737). Invented an instrument for trisecting an angle.

Chuquet. Wrote in 1484, in French, an arithmetic, LE TRIPARTY.

de Moivre (Abraham de Moivre; *b.* Vitry, 1667; *d.* London, 1754). Came to England as a Huguenot refugee on the Revocation of the Edict of Nantes, 1685; a friend of Newton and Halley and a member of the Royal Society; an originator of the application of trigonometry to complex quantities. It is told of him that, being ordered by his doctor to increase his hours of sleep by a certain amount daily, he followed the advice with the precision of an arithmetical progression until he died of "not waking."

Descartes (Réné Descartes du Perron; *b.* near Tours, 1596; *d.* Stockholm, 1650). On military service in the Netherlands, he was attracted by a placard in Dutch in the streets of Breda. He asked a passer-by, who happened to be the Principal of the College of Dort, to translate it into French or Latin; the latter agreed to do so if Descartes would answer it. It was a geometrical problem with a challenge to the world to solve it. Descartes succeeded in a few hours. This incident revealed to him his powers. He is chiefly known for his philosophy and for the invention of the analytical method of geometry—called after him, "Cartesian Geometry." Weakly, and of small stature, he was nevertheless of great courage. Once when he overheard a plot of some sailors to kill and rob him he attacked them with such fury as thoroughly to intimidate them.

Diophantus (Alexandria; third century A.D.). His syncopated algebra is the first treatise on the Solution of Equations.

Eratosthenes (*b.* Cyrene, 275 B.C.; *d.* Alexandria, 194 B.C.). An Alexandrian astronomer; his chief mathematical work was the investigation of primes by means of the SIEVE. As an astronomer he measured the obliquity of the Ecliptic and made a determination of the Earth's circumference.

Euclid (third century B.C.). The first Lecturer in Mathematics at the Museum (Gk. *Mousaion, the Home of the Muses*), the university of the ancient world, founded by Ptolemy Soter. His Elements collected the discoveries of the Pythagoreans into a deductive system of geometry, culminating in the discussion of properties of the regular solids. They form the foundation of geometrical teaching to this day. (*See* pp. 21, 23, 27.)

Euler (Leonhard Euler; *b.* Basel, 1707; *d.* St. Petersburg, 1783). Brought under systematic treatment the mathematical discoveries of the golden age of Newton, Leibnitz, etc., and gave them to the world in an ordered form, introducing suitable notations and supplying proofs. The service he rendered to modern mathematics was of much the same kind as Euclid rendered to Greek

geometry. He lost the sight of one eye in 1735 and of the other in 1768; but in spite of blindness he pursued with unabated energy his work, which totalled an output of nearly eighty quarto volumes. His memory was prodigious: he could repeat the whole of the Æneid, and when blind performed, mentally, elaborate computations.

Fermat (Pierre de Fermat; *b.* near Montauban, 1601; *d.* Castres, 1665). The greatest investigator in the Theory of Numbers; his methods are not known. Of the theorems that bear his name the most celebrated is that which says that $a^n + b^n = c^n$ is incapable of solution with integral values of a, b, and c if $n > 2$.

Fibonacci. *See* Leonardo.

Galileo (Galileo Galilei; *b.* Pisa, 1564; *d.* near Florence, 1642). Was too great a man to be dismissed in the few lines that can be given to him here. For an account of his astronomical work and his contributions to Dynamics, into which he introduced experimental methods of investigation, a work of Biography or Lodge's PIONEERS OF SCIENCE should be consulted. Old, blind, and imprisoned, he was visited by Milton who refers to him in the lines:

"The moon whose orb
Through optic glass the Tuscan artist views."—*P.L.* 288.

Gauss (Karl Friedrich Gauss; *b.* Brunswick, 1777; *d.* Göttingen, 1855). The greatest mathematician of his day.

Gregory (James Gregory; *b.* Aberdeenshire, 1638; *d.* Edinburgh, 1675). Professor of Mathematics at St. Andrews and Edinburgh; established the series from which all subsequent evaluations of π have been directly or indirectly made.

Gulden (Habbakuk Gulden, or Guldinus; *b.* St. Gall, 1577; *d.* Grätz, 1643). Published theorems on the surface-area and volume of solids of revolution, theorems due to Pappus.

Halley (Edmund Halley; *b.* Haggerston, 1656; *d.* Greenwich, 1742). Chiefly known for his calculation of the paths of comets, he was largely instrumental in causing Newton to publish his *Principia*, as Newton himself acknowledges in his preface. He published the first complete Mortality Tables. At his suggestion Sharp, in 1699, made the first computation of π by the modern method.

Harriott (*b.* Oxford, 1560; *d.* London, 1621). Spent the early part of his life in America with Sir Walter Raleigh making surveys. He was the first to show that $f(x) = 0$ can be solved by using the factors of $f(x)$.

Hero (or Heron, of Alexandria; about 80 B.C.). Established the formula for the area of a triangle in terms of its sides; invented a steam-engine, a theodolite, and other surveying instruments.

Hipparchus (*b.* Nicæa in Bithynia, 183 (?) B.C.; *d.* Rhodes, 125 (?) B.C.). A great astronomer and the originator of trigonometry; he measured angles by the chords subtending them in a circle,

the angles being at the centre. He constructed the first trigonometrical table—a table of chords. He originated the use of latitude and longitude for fixing geographical position. Delambre considered him "one of the most astonishing men of antiquity."

Hippocrates (of Chios ; *b.* about 470 B.C.). Wrote a textbook of geometry on which Euclid's Elements may have been founded, using letters to mark points in his diagrams and introducing the *reductio ad absurdum* method. He was the first to determine a curvilinear area, that of a lune. He is not the celebrated physician, but was his contemporary.

Huygens (Christian Huygens v. Zuylichen ; *b.* The Hague, 1629 ; *d.* The Hague, 1695). His work is comparable in power and extent with that of Newton, with whom he was contemporary, and whose acquaintance he made, coming to London to do so. He had previously met Leibnitz in Paris.

Kepler (Johann Kepler ; *b.* near Stuttgart, 1571 ; *d.* Ratisbon, 1630). An astronomer who did work in mathematics which involved the ideas of the infinitesimal calculus ; helped to introduce logarithms.

Krämer or Kremer. *See* Mercator.

Lami (Bernard Lami ; *b.* Le Mans, 1640 ; *d.* Paris, 1715). A priest ; author of *Traité de Méchanique*, 1687.

Leibnitz (Gottfried Wilhelm Leibnitz ; *b.* Leipzig, 1646 ; *d.* Hanover, 1716). One of the mathematicians for whom is made the claim to priority in inventing the Calculus. For part of his life he had George I, then Elector of Hanover, as his patron ; but on the latter's accession to the throne of England, Leibnitz was neglected.

Leonardo (Leonardo Fibonacci of Pisa ; *b.* about 1175). Was educated in Barbary, and travelled extensively. His LIBER ABACI introduced the Arabic system of numeration into Europe ; he solved a cubic correct to ten significant figures.

Leonardo da Vinci. *See* da Vinci (p. 16).

Lobachewski (Nicholas Lobachewski; Kasan; *b.* 1793; *d.* 1856). Developed the geometry in which the angle-sum of a triangle may be less than two right angles.

Machin (John Machin ; *d.* 1751). Secretary of the Royal Society, 1718–1747.

Mascheroni (Lorenzo Mascheroni ; *b.* Castagneta, 1750 ; *d.* Paris, 1800). Elaborated a method of geometry in which the straight line is not needed for constructions.

Menæchmus (375–325 B.C.). A tutor of Alexander the Great ; the first to discuss conic sections.

Menelaus (of Alexandria). Published, about A.D. 98, a work on spherical trigonometry.

Mercator (Gerhard Kremer, or Krämer ; *b.* Rupelmonde, 1512 ; *d.* Duisburg, 1594). A great cartographer.

Moivre. *See* de Moivre.

Müller. *See* Regiomontanus.

Napier (John Napier of Merchiston; *b.* 1550; *d.* 1617). Invented and compiled tables of logarithms. Kepler's warm eulogy of Napier's work prefaced his *Ephemeris* for 1620.

Newton (Isaac Newton; *b.* Woolsthorpe, near Grantham, 1642; *d.* Kensington, 1727). The greatest mathematical genius the world has known; we cannot here even summarize his stupendous achievements nor indicate the extent and variety of his powers. Of him Leibnitz said that, taking mathematicians from the beginning of the world to the time when Sir Isaac lived, what he had done was much the better half; that he himself had consulted all the learned in Europe on some difficult points without having any satisfaction, and when he applied to Sir Isaac, he wrote him in answer by the first post to do so-and-so, and then he would find it. Of himself, when an old man full of years and honour, he said, "I know not what the world will think of my labours, but to myself it seems that I have been but as a child playing on the seashore; now finding some pebble rather more polished and now some shell rather more agreeably variegated than another, while the immense ocean of truth extended itself unexplored before me."

Oughtred (William Oughtred; *b.* Eton, 1575; *d.* Albury, in Surrey, 1660).

Pappus (i.e. *Papa*). Lived at Alexandria about A.D. 300, and wrote a work Συναγωγή (the same word as synagogue, meaning *assemblage*), containing all the mathematical knowledge of his time.

Pascal (Blaise Pascal; *b.* Clermont, 1623; *d.* Paris, 1662). A precocious mathematical genius, that his own feeble health and his father's wishes could not prevent from following his bent. His correspondence with Fermat laid down the principles of the Theory of Probability. In 1654, persuaded by what he considered a Divine summons to abandon the world, he devoted the rest of his life to religion. His LETTRES PROVINCIALES and his PENSÉES are well known.

Playfair (John Playfair; *b.* near Dundee, 1748; *d.* Edinburgh, 1819).

Ptolemy (Ptolemæus Claudius; *b.* Ptolemais, A.D. 87; *d.* Alexandria, 168). The author of the ALMAGEST, a title said to be an Arabic corruption of μεγίστη σύνταξις or *great treatise.* This, the great Greek work on astronomy and trigonometry, was founded on the writings of Hipparchus.

Purbach (Georg Purbach, or Peurbach; *b.* near Linz, 1423; *d.* Vienna, 1461). Tutor of, and collaborator with, Regiomontanus, he helped to lay the foundations of modern trigonometry.

Pythagoras (*b.* Samos, 570 B.C.; *d.* Metapontum 501 B.C.). He and his school investigated much of the geometry, plane and solid, that forms the substance of Euclid's Elements; in philosophy known for his belief in metempsychosis. (*See* Chapter VI.)

Ramus (Pierre de la Ramée ; *b.* in Picardy, 1515 ; killed in the Massacre of St. Bartholomew, 1572). Led a revolt against the authority of Aristotle and introduced a more modern treatment of Euclid.

Recorde (Robert Recorde ; *b.* Tenby, 1510 ; *d.* London, 1558). Physician to Edward VI and Mary ; author of THE WHETSTONE OF WITTE.

Regiomontanus (Johannes Müller ; *b.* Königsberg, 1436 ; *d.* Rome, 1476). Author of the first modern Trigonometry.

Riemann (Georg Friedrich Bernhard Riemann ; *b.* Breselenz, 1826 ; *d.* Selasca, 1866). Developed the geometry in which the angle-sum of a triangle may be greater than two right angles.

Rutherford (William Rutherford ; *b.* 1798 (?) ; *d.* Charlton, S.E., 1871). A Scotsman, he became lecturer at the R.M.A., Woolwich.

Roberval (Gilles Personier de Roberval ; *b.* Roberval, 1602 ; *d.* Paris, 1675).

Saccheri (Girolamo Saccheri). Published, in 1733, at Milan, EUCLIDES AB OMNI NÆVO VINDICATUS, in which, after showing the possibility of non-Euclidean geometries, he shirked his own conclusions.

Simpson (Thomas Simpson). Published AREAS OF CURVES, ETC., BY APPROXIMATION, 1743.

Simson (Robert Simson ; *b.* Ayrshire, 1687 ; *d.* 1768). Devoted himself to critical research in the Greek geometers, Euclid and Apollonius. His EUCLID'S ELEMENTS (published at Glasgow in 1756) is the basis of modern English textbooks. He ascribed all faults in Euclid to his editors and commentators.

Snell (Willebrord Snell van Roien, 1581–1626). Professor at Leyden ; the most considerable trigonometrist of his day ; especially famous for his investigation of the survey problem of Resection from Three Points.

Stevin (Simon Stevin, or Stevinus ; *b.* Bruges, 1548 ; *d.* Leyden or The Hague, 1620). A pioneer in the study of mechanics.

Stifel (Michael Stifel ; *b.* Esslingen, 1486 ; *d.* Jena, 1567). An Augustine monk who became a Lutheran ; predicted that the world would come to an end October 3, 1533, and in other ways lent his powers to fanciful interpretation of the Revelation.

Tartaglia (Niccola Fontana, known as Tartaglia *the Stammerer* ; *b.* Brescia, 1500 ; *d.* Venice, 1557). The impediment in his speech was caused by having his palate cleft in a massacre when the French took Brescia, 1512 ; his treatise on arithmetic is one of the chief authorities for our knowledge of Italian mathematics of his time. Some of his examples figure prominently in present-day puzzle books.

Thales (of Miletus, *b.* 624 B.C. ; *d.* Athens, 546 B.C.). The FATHER OF GEOMETRY and one of the "Seven Wise Men." A merchant who travelled widely, and learning the knowledge of

the Egyptians, started the study of Abstract Geometry. The story is told of him, that on a trade journey one of his donkeys laden with salt slipped in a stream, and finding that his load had become lighter, he formed a habit of rolling over in streams until Thales cured him by loading him with sponges.

Torricelli (Evangelista Torricelli ; *b*. Faënza, 1608 ; *d*. Florence, 1647). Wrote on areas and lengths of curves, and mechanics.

Vieta (François Viète ; *b*. Fontenay, near La Rochelle, 1540 ; *d*. Paris, 1603). To him belongs the chief credit for the introduction of Symbolical Algebra.

Wallis (*b*. Ashford, 1616 ; *d*. Oxford, 1703). Introduced the use of infinite series into analysis ; the first to obtain an expression for π whose form followed a simple law ; in a competition with Wren and Huygens on the collision of bodies, he produced the most complete treatment. He was the first to devise teaching for deaf mutes.

Widmann (Johannes Widmann von Egger ; *b*. about 1460). His mercantile arithmetic, published at Leipzig in 1489, first uses + and −.

Galileo, Halley, and Kepler, known chiefly as astronomers, have been included in the catalogue of mathematicians ; so has Descartes, who is the author of a system of philosophy. The catalogue may be extended to include some artists, politicians, and some names known in literature.

Dürer (Albrecht Dürer ; *b*. Nürnberg, 1471 ; *d*. 1528). A great engraver, in whose engraving of Melancholia can be seen a magic square. He gave an approximate construction for a regular pentagon thus :

Let ABC be an equilateral triangle ; with centres A, C, B and radius AB circles are drawn.

Circles with centres A and C meet in F. G is the mid-point of the minor arc AB of circle of centre C. FG produced meets circle of centre B at H.

The angle ABH is approximately the angle of the pentagon.

He also showed that the side of a regular pentagon is nearly half the side of an equilateral triangle of equal area.

da Vinci (Leonardo da Vinci ; Florentine ; *b*. 1452 ; *d*. 1519). Painter of the "Last Supper," and "La Gioconda." He was also a poet. He left a number of notebooks in which the mechanics of machines is discussed, the principle of moments recognized, and a design for a flying machine is given. An ingenious proof of Pythagoras' Theorem is attributed to him ; he was the first to explain earth-shine on the moon. One of the greatest thinkers the world has produced, he is credited with anticipating many modern inventions. (Read Merejowski's Forerunner, a novel of which he is the hero.)

Wren (Sir Christopher Wren ; *b*. Knoyle, Wilts, 1632 ;

d. London, 1723). Savilian and Gresham Fellow of Astronomy, President of the Royal Society; a friend of Halley and Newton, he made notable contributions to mathematical progress.

In literature there may be mentioned:

Bede (The Venerable Bede; *b*. near Jarrow, 672; *d*. 735). Wrote on Arithmetic.

Ben Ezra (Rabbi Abraham Ben Ezra; *b*. Toledo, 1097; *d*. Rome, 1167), who is mentioned in Browning's "Holy Cross Day" and is the subject of Browning's "Rabbi Ben Ezra," helped to introduce Moorish learning into Europe. In his arithmetic he explains the Arabic System of Numeration.

Boëthius (*b*. Rome, about 475; *d*. Ticinum, 526). Author of the *Consolations of Philosophy*, which King Alfred translated, was the chief scholastic figure of the Dark Ages; his geometry, which contained little more than the Enunciations of Euclid, was the textbook for many centuries. He was tortured and put to death for his integrity after a life of prominence and philanthropy. He is Chaucer's Bœce, and Chaucer draws freely on his *Consolations* and often refers to him.

In the Nonne Priestes Tale:

> "But I ne can not bult it to the bren
> As can the holy doctour Augustyn
> Or Bœce or the bishop Bradwardyn."

And again

> "Therwith ye han in musik more felinge
> Than hadde Bœce or any that can singe."

Lewis Carroll. "Lewis Carroll," the *nom de plume* of Charles Lutwidge Dodgson (1832–1879), is derived from his Christian names. He was a don of Christ Church, Oxford, and the author of some mathematical works, of which the most notable is *Euclid and his Modern Rivals* (1879). There is a story that Queen Victoria, delighted with *Alice in Wonderland*, ordered copies of all his works, and was a little surprised on receiving a number of mathematical works.

Hypatia (murdered in A.D. 415, as related in Kingsley's novel). Was the last Alexandrian mathematician of note. She was the daughter of Theon, whose commentated version of Euclid was the foundation of subsequent editions.

Omar Khayyám (*d*. Naishapur in Khorassan, 1123). Author of the *Rubáiyát*. Was a Persian astronomer and algebraist; he extracted square roots and solved many cubic equations; helped to reform the calendar.

Pascal. (*See* above, p. 14.)

Plato (Athens, 429–348 B.C.). A pupil of Socrates, whose dialogues he recorded; lectured in the gymnasium of the Academeia, and by his influence helped to make Greek geometry an abstract science.

A few politicians of repute have also been mathematicians:

Carnot (Lazare Nicolas Marguerite Carnot; *b.* Burgundy, 1753; *d.* in exile, 1823). An enthusiastic republican; he had been educated at a military school and performed the functions of general staff and was the "Organizer of Victory" in the early campaigns of the Republic. One of the first members of the newly founded Institut, he was succeeded by Napoleon. He was the author of many important mathematical works.

Condorcet (Marie Jean Antoine Nicolas Caritat, Marquis de Condorcet; *b.* Ribemont, near St. Quentin, 1743; *d.* Bourg la Reine, 1794). A mathematician of European reputation. He took an active part in the French Revolution, but his moderation caused him to be outlawed. After long hiding he was taken, and to escape the executioner took poison.

de Witt (Jean de Witt; *b.* Dordrecht, 1628; *d.* The Hague, 1672). A democrat and opponent of the House of Orange, he became Raad Pensionarius ("Grand Pensioner") in 1652 and organized the Dutch marine against England during the Commonwealth and the reign of Charles II. When Louis XIV invaded Holland, William of Orange, subsequently King of England, was made Stadtholder, and de Witt and his brother were put to death by the populace. He was a pupil of Descartes, and the first to apply the Theory of Probability to Economic Science.

Painlevé, the French Premier during part of the Great War, is a mathematician of repute.

CHAPTER II

HISTORICAL OUTLINE

Beginnings.—The early history of mathematics, like that of politics, is lacking in records. Of the first gropings we can only conjecture; its early developments come to us through tradition.

If we may judge by the condition of knowledge among primitive peoples, we shall assume that the first and most rudimentary mathematics was counting, and that it must have needed many centuries for the human race to learn to count even as far as 5. We may conjecture that geometry had its beginnings only when men ceased to be nomadic and, in settling on the land, required some system of land-apportioning or surveying. We may assume that the watchers of the stars, shepherds and sailors, originated astronomy.

It is difficult for us, accustomed from our early days to familiarity with the names and signification of the numerals and with fairly accurate ideas of weighing and measuring, to realize how slowly the human race developed a number-system and means of measurement. It is still harder to realize how great were the contributions to human thought and progress made by the first mathematical thinkers.

But it becomes easier if we remember that there are even now tribes who cannot count beyond 5; that till well within historical times land was measured by the amount an ox could plough in a day, and that our standards have been comparatively recently evolved from such variable units as the foot, the stone, etc.

Arithmetic probably originated with the commercial needs of a great trading people, such as the Phœnicians; geometry with the necessity, in such a country as Egypt, of ensuring to each landowner his right amount of land; astronomy with the sailors' use of the stars for navigation. But the first results and rules would be empirical and isolated; the first formulæ would be tentative and obtained inductively.

Such appears to be the case in the first mathematical work of which we have any record. It is the RHIND PAPYRUS, in the British Museum, a copy of a work of Ahmes entitled DIRECTIONS FOR KNOWING ALL DARK THINGS, written about 1700 B.C. But the original may be a thousand years older.

It contained some rules for fractions, for simple mensuration, some solutions of simple equations, and, as far as can be judged, a little very rudimentary trigonometry connected with the dimensions of pyramids.

Here is a typical algebraical problem: "*Heap;* its two-thirds, its half, its seventh, its whole make* 33." The solution is given as $14 + \frac{1}{4} + \frac{1}{97} + \frac{1}{56} + \frac{1}{679} + \frac{1}{776} + \frac{1}{194} + \frac{1}{388}$.

No method is given, and it is generally believed that the solution was obtained empirically. If a method had been known, it should have produced the simpler form, $14 + \frac{1}{4} + \frac{1}{28} + \frac{1}{5044}$, Egyptian fractions being restricted to those with unit numerator; but in any case the result indicates considerable manipulative skill.

The Early Greeks.—While mathematics was directed to utilitarian ends, its methods remained inductive and progress was slow, probably confining itself to achieving the utilitarian result. It is to the intellectual curiosity of the Greek mind that we owe the introduction of deductive methods and the rapid development of mathematics, and, in particular, of geometry, as an ordered science. Contact with the knowledge of the ancient civilizations of Asia and Egypt aroused the curiosity of travelled Greeks, and there was produced a line of philosophers, who founded schools in which investigation and teaching accumulated and handed on the expanding volume of knowledge.

These schools (Gk. *Schole, leisure*) were associations of men of leisure, devoted to the search for truth and to the discovery of the obscure; the relation of master and disciple was not so much that of a modern teacher and his class, as that of one of the great painters, such as Rubens or Murillo and his "school," and consequently, as was particularly the case with the Pythagoreans, it is often difficult to separate the work of the master as an individual from that of the school as a society.

The first was THALES of Miletus (624–546 B.C.), who started the study of geometry as a deductive science. He is credited with knowing that the circle and the isosceles triangle were symmetrical figures, that the angle in a semicircle is a right angle, that the sides of equiangular triangles are proportional, that the observation of a distant point is determined by a base line and two angles—the fundamental proposition of range-finding and surveying.

PYTHAGORAS (570–501 B.C.) was a pupil of a pupil of Thales; he and his school discovered most of the geometry, the theory of numbers, the summation of series which form the content of Euclid's Elements. Their influence and teaching spread throughout the Greek world.

So far geometry had been the geometry of the straight line and circle; and constructions limited to the use of these proving

* "Heap" was used for the unknown quantity in Egyptian problems.

inadequate for the solution of certain problems (*see* Chaps. III and IV), other curves, particularly the Conic Sections, were discovered and their properties investigated.

The Alexandrians.—Alexander the Great had built at the mouth of the Nile the city of Alexandria. Here, after his death, Ptolemy Soter, who became ruler of Egypt, founded the first university. It was called the MOUSAION (of which the Latinized form is MUSEUM), or *The Home of the Muses.* Here for many centuries was the focus of the world's intellectual activity—the centre to which great thinkers came to study, the centre from which Greek philosophy radiated.

The first lecturer in Mathematics was EUCLID. His ELEMENTS (about 300 B.C.) put into final form the geometry of the straight line and circle, including the geometry of the regular solids.

APOLLONIUS (260–200 B.C.) did the same for the geometry of the Conic Sections.

ARCHIMEDES (287–212 B.C.) produced works of the greatest importance on mechanics and hydrostatics, as well as essays on various geometrical problems.

ERATOSTHENES (275–194 B.C.), HIPPARCHUS (183–125 B.C.), and PTOLEMY (A.D. 87–168) laid the foundations of mathematical astronomy; HIPPARCHUS, HERO (about 80 B.C.), and PTOLEMY of trigonometry; HERO of mechanics, and DIOPHANTUS (A.D. third century) of algebra.

Under the Roman Empire the glories of Alexandria faded: her mathematicians became commentators instead of discoverers; she marked time instead of going forward. But while Europe sank into the intellectual gloom of the Dark Ages, Alexandria remained the treasure-house of Greek culture and was eventually destined to be the source of a new impulse of intellectual illumination.

The Moors.—In A.D. 641 Alexandria, with its library of precious books, was taken by the Mohammedans. Their empire rapidly spread to India in the east and to the Pyrenees in the west; and they became the heirs of the Greek mathematicians, directly by becoming possessors of the Library of Alexandria, and indirectly by coming into contact with Hindu mathematics. With the rapid growth of empire, there was a renaissance of the spirit of scientific inquiry. In everything progress was quickened; in medicine and architecture, in physics and mathematics, in chemistry and astronomy new ideas and impulses sprang into being. Wherever their conquests reached, universities, liberally endowed, were founded.

While the mathematical teaching of Western Europe was confined to the geometry of Boëthius, consisting of little more than enunciations of Euclid's propositions, the Moors were translating and studying the texts of the great Greeks, developing

a method of arithmetic with the Arabic numerals imported from India, using the Algebra of ALKHWARIZMI, which probably had Diophantus for its remote ancestor, and laying the foundations of the study of optics.

Words in common use to this day testify to the width of their dominion and the extent of their sciences, such as, alchemy, alcohol, algebra (*see* Alkhwarizmi, p. 9), alhambra, alkali, almagest (*see* Ptolemy), admiral (Fr. *amiral* from *emir*), Guadalquivir (*wady*).

The Middle Ages.—This revival of learning eventually reached Europe. Euclid's geometry was introduced, about A.D. 1120, by ADELHARD, who, disguised as a dervish, studied at Cordova University and learnt Greek geometry from Arabic texts. LEONARDO of Pisa, trading in Barbary, brought back the Arabic arithmetic and algebra ; and the Emperor Frederick II (*b.* 1194) employed Jews to obtain and translate the Arabic texts of Apollonius, Archimedes, Aristotle, etc., and so gave to Europe once more the learning of the Greeks. This was chiefly geometry, but, in the centuries which followed, arithmetic, algebra, and trigonometry took shape ; symbolism developed both in arithmetic and algebra, and in algebra the cubic equation was definitely solved.

The Age of Newton.—Even so, if we except the geometry of curves other than the circle, we shall find that the sum total of mathematical knowledge at the beginning of the seventeenth century would make a modest curriculum for the average modern schoolboy. But the Renaissance had already made itself felt. Europe had emerged from the Dark Ages into light, and was striving for more light. The world was ready for great ideas and discoveries. And a period of astounding brilliance ensued : NEWTON (1642–1727), GALILEO (1564–1642), LEIBNITZ (1646–1716), PASCAL (1623–1662), DESCARTES (1596–1650), HUYGENS (1629–1695), HALLEY (1656–1742) are some of the greatest luminaries—Newton the greatest of all. In a few years the horizon of mathematics had expanded inconceivably : the algebra of series and complex number, the differential calculus, algebraic geometry, mathematical mechanics, and astronomy were all invented or put on to their modern footing.

There followed in the eighteenth century EULER (1707–1783), who put all this amazing mass of discovery into textbook form, with suitable notation, proofs, and additions ; thus rendering to the teaching and spread of mathematics one of the greatest services it has ever received.

This is the merest outline of the history of Elementary Mathematics. We add a few words on separate divisions of the subject :

Geometry.—The word means *land measuring*, and, as the name implies, the subject probably had its origin in surveying, knowledge indispensable in a country where landmarks were

obliterated by annual floods. The Greeks early developed the subject, many of their first discoveries being doubtless due to reflection on the properties of figures, the shapes of patterns in tiles; but they pushed their geometry to a point where it dealt not only with the properties of figures, but was made to sum series of numbers, to solve problems that lead to quadratic equations, and to build up the theory of proportion in such a way as to include irrational numbers. All these are found in Euclid's Elements.

Little apparently was left to do in geometry. But in the seventeenth century DESCARTES, by his invention of co-ordinates, reduced the treatment of geometry to algebraical methods. And in the nineteenth century Projective Geometry, which owes its invention to DESARGUES, a friend of Descartes, and the non-Euclidean geometries were elaborated.

Algebra.—The first systematic algebra is that of DIOPHANTUS. Little is known about him except that he was alive in Alexandria in the first half of the fourth century A.D. Problems had been solved before his time by algebraic reasoning, but without the use of a symbolical notation. Diophantus uses initials and abbreviations of words as symbols—a treatment termed "syncopated algebra." He solves simultaneous equations by the process of elimination, the general quadratic, some indeterminate equations, and one cubic.

He lived too late to influence Greek mathematics, but his work is said to have been taken to India and developed there, and so indirectly, through the Moors, he did influence the mediaeval mathematics of Europe.

LEONARDO'S LIBER ABACI, published in 1202 as *Algebra et Almuchabala*, solves simple, quadratic, and indeterminate equations, without using either symbols or abbreviations, i.e. it is rhetorical algebra.

Symbols were gradually invented and accepted, and by the end of the sixteenth century the algebra of the solution of equations (where the solution was real) was practically complete. CARDAN and TARTAGLIA had solved the general cubic; FERRARI had reduced the solution of the quartic or biquadratic to that of the cubic; and VIETA had laid the foundations of the modern symbolism of algebra and had devised a method for solving approximately any algebraical equation.

In the seventeenth century the Binomial Theorem was discovered as a general theorem, PASCAL and NEWTON having a share in it; and the algebra of infinite series was evolved.

It is to be especially remarked that the development of algebra was stimulated by, and continued progress depended and still depends on, the invention of convenient symbols.

Meanwhile the solution of equations of the fifth and higher degree had not been found, and it was not till ABEL (1802-1829)

showed the impossibility of solving a general case that the attempt was abandoned.

Arithmetic.—PYTHAGORAS and PLATO paid a good deal of attention to the Theory of Numbers and wove various fanciful ideas around them. As mathematical theory, it was a question of determining primes, perfect numbers (a perfect number is one which is the sum of its factors, e.g. 6 and 28), and amicable numbers (a pair of numbers are amicable when each is the sum of the factors of the other).

The great modern investigators of properties of numbers are MERSENNE, a Franciscan friar (*b.* 1588; *d.* Paris, 1648), and FERMAT (1601–1665).

Modern computational arithmetic depends for the convenience and rapidity of its method on the introduction of the Arabic numerals and its more recent developments on the introduction of contracted methods in decimals.

Trigonometry.—Trigonometry has its origin in surveying, and the application of mathematics to astronomy. The Rhind Papyrus implies the use of a ratio of sides of a right-angled triangle for the construction of pyramids to satisfy certain linear conditions; but the subject owes its real inception to HIPPARCHUS (*b.* 183 B.C.). Geometry gives us certain relations of sides and angles of a triangle, but does not connect by definite laws the lengths of the sides with the magnitude of the angles; nor does it give any means of connecting the magnitude of an angle with the length of a line-segment. HIPPARCHUS, by computing a table of chords which subtend angles of different magnitude at the centre of a circle of standard radius, laid the foundation of trigonometry. His work in astronomy and trigonometry is known to us through the medium of PTOLEMY'S ALMAGEST, written three centuries later. From Ptolemy's Theorem (Euclid VI D.), that *the rectangle contained by the diagonals of a cyclic quadrilateral is equal to the sum of the rectangles contained by pairs of opposite sides*, can be deduced directly as special cases the important formulæ for $\sin (A \pm B)$; $\cos (A \pm B)$.

Trigonometrical methods dependent on the use of a table of chords, suitable as they are for the geodesical and astronomical problems that gave rise to the subject of trigonometry, did not prove suitable for the analytical development of Plane Trigonometry. The invention of **sine** and **cosine** derived from Arabic sources by PURBACH and REGIOMONTANUS, A.D. 1464, proved more convenient. At first, sine, cosine, etc. (as the terms "tangent" and "secant" especially remind us), were originally lengths of lines connected with a circle of standard radius. EULER (1707–1783), by using them for ratios and by introducing the notation **a, b, c, A, B, C** for the magnitudes of the sides and angles of a triangle, advanced the development of the subject another step.

De Moivre's Theorem makes trigonometry available for the Theory of Complex Numbers and those parts of algebra that depend on it. Thus trigonometry, using elementary algebraical methods for geometrical purposes, repays its debt by a contribution to the treatment of questions in higher algebra.

Mechanics.—The Science of Machines was first submitted to mathematical treatment by Archimedes (287–212 B.C.). He dealt with the lever, and used the mode of treatment which Euclid used in his Elements, making from certain assumptions a deductive system of treatment of the lever and centre of gravity. To build up mechanics from intuitive assumption was also attempted by Stevin (1548–1620), who started with the assumption that an endless chain slung over a wedge is in equilibrium. His work Wiscontige gedachtnissen, published at Leyden in 1605, was translated by Willebrord Snell into Latin under the title *Hypomnemata Mathematica.*

Others attempted later on to prove the parallelogram of forces. As mechanics is partly a natural science and must rest on a foundation of experiment or intuition, all these attempts were in some respect unsatisfactory.

Of the three fundamental principles, those of (1) the lever, (2) the inclined plane, (3) the parallelogram of forces, it is sufficient to assume one, for from any one the others can be deduced.

In present-day teaching two principles are usually established by experiment and intuition, viz., the principles of the lever and of the parallelogram of forces. Thus established, they serve as the fundamental assumptions of the mathematical treatment.

Newton's Laws of Motion are hypotheses to account for phenomena, and were submitted by him to a test of verification from the moon's motion. His first attempt at verification failed. One of the factors in the test was the length of the earth's diameter, and when a new determination of this had been made, he returned to his test and obtained a convincing result. They are thus scientific hypotheses rather than mathematical certainties; but their substantial truth is confirmed by three centuries of astronomical prediction that has been confirmed by the event.

Being hypotheses, they may require modification to conform to the advance in other branches of mathematics and physics.

Leonardo da Vinci (1452–1519) recognized the importance of statical moments, and, accepting this principle, Galileo deduced the Principle of the Inclined Plane.

Galileo (1564–1642) investigated the Laws of Falling Bodies and of simple pendulums. Making suitable assumptions, he submitted them to logical inquiry and experiment, and so laid the foundation of Dynamics.

Wallis, Wren, Huygens in December, 1668, and January, 1669, at the invitation of the Royal Society, investigated the impact of inelastic and elastic bodies.

But it is to NEWTON'S imagination and achievements that the science of mechancis is the greatest debtor. His Laws of Motion made it possible to treat mechanics as a deductive science like geometry, and to apply geometrical methods; while his Law of Universal Gravitation extends the application of the science to the whole universe.

During the Middle Ages and until the end of the seventeenth century Latin was the *lingua franca* of the mathematician as of the politician. He wrote his works in Latin; he announced his results in Latin rebuses or anagrams in order to vindicate his claim to priority of discovery; he carried on disputations in Latin. These disputations and challenges issued by discoverers of new methods promoted mathematical activity and intercourse.

We find a Minorite Monk about 1240 writing the CARMEN DE ALGORISMO, in which he derives the word algorism from the name of a Hindu king. It begins (but employing the abbreviations usual in MS.).—

"Haec algorismi ars praesens dicitur in qua talibus indorum friumur bis quinque figuris," and is translated:

"This boke is called the boke of algorism or augrym after lewder use. And this boke tretys of the craft of nombryng, the quych crafte is called algorym. Ther was a king of Inde the quich heyth (= is named) Algor and he made this craft algorism, in the quych we use ten figurys of Ind."

And Rouse Ball quotes from a work of Pacioli published in Venice in 1494, the solution of a quadratic that we should write $x^2 + x = a$:

"Si res et census numero coequantur, a rebus dimidio sumpto censum producere debes, addere numero, cujus a radice totiens tolle semis rerum census latusque redibit." [*res* means the unknown; *census* its square.]

The first case of the challenge of which we have a record occurs when Frederick II held a mathematical tournament at Pisa in 1225, in order to test Leonardo's skill; it was at this contest that the latter gave the solution of a cubic to 10-figure accuracy, and at the same time proved that it could not be of the form $\sqrt{\sqrt{m} + \sqrt{n}}$

In 1530, when Tartaglia announced that he could solve equations of the type $x^3 + px^2 = r$, Fiore, doubting him, challenged him to a contest. Each was to propound thirty problems to the other, and the winner was the one who solved most in thirty days. Tartaglia, knowing that Fiore could solve empirically equations of the type $x^3 + qx = r$, set to work to find a general solution, and was so successful that he solved Fiore's problems, which were of the kind he anticipated, within two hours.

The solution to a biquadratic obtained by Ferrari was the consequence of a challenge to mathematicians to solve $x^4 + 6x^2 + 36 = 60x$.

Descartes' decision to devote himself to mathematics was greatly influenced by a challenge he saw posted in Breda.

As examples of the cautious announcement of discovery we have Huygens' 7 *a*'s, 5 *c*'s, 1 *d*, 5 *e*'s, 1 *g*, 1 *h*, 7 *i*'s, 4 *l*'s, 2 *m*'s, 9 *n*'s, 4 *o*'s, 2 *p*'s 1 *q*, 2 *r*'s, 1 *s*, 5 *t*'s, 5 *u*'s. It would be difficult enough to form the right sentence, even with some clue to its significance; without the clue, who would find in it a statement of the discovery that the apparent excrescences of Saturn were a ring? Thus, *annulo cingitur tenui, plano, nusquam cohaerente, ad eclipticam inclinato.*

The discovery of these excrescences had been one of the first fruits of Galileo's invention of the telescope, and he had announced the discovery in letters which, transposed, read *altissimum planetum tergeminum observavi.*

In the same way, in a letter to Leibnitz, Newton announced his discovery of Fluxions (i.e. the Differential Calculus) by the sentence, transposed, *Data æquatione quot-cunque Fluentes quantitates involvente, Fluxiones invenire, et vice versa.*

CHAPTER III

SOME CONSIDERATIONS OF EUCLID'S GEOMETRY

The Greeks aimed at making their geometry part of a science of abstract thought. To start with a minimum of premisses, and those such as could be readily accepted even by a philosophic mind, and to proceed by logical deduction from these until the fabric of geometry was complete; this is the principle on which Euclid's Elements are compiled. Intuition and, to a greater degree, conclusions drawn from practical measurements were recognized as fallible.

Plutarch says, "Plato blamed Eudoxus, Archytas, and Menæchmus and their school for endeavouring to reduce the duplication of the cube to instrumental and mechanical contrivances; for in this way (he said) the whole good of geometry is destroyed and perverted, since it backslides into the things of sense, and does not soar and try to grasp eternal and incorporeal images; through the contemplation of which God is ever God."

The progress of Greek geometry from Plato onwards, directed by Plato's influence, follows Plato's ideal; and Euclid's Elements, compiled one hundred years later, is an attempt to make the whole of geometry conform to it. Starting with a certain number of definitions, postulates, and axioms, he built up by logical deduction his geometrical system. Though not irreproachable, it was such a remarkable achievement that for many centuries it commanded the loyal veneration of mathematicians and schoolmen, and in the light of modern thought is recognized as having dealt in a masterly way with the greatest difficulties. In France, Ramus (Pierre de la Ramée, A.D. 1515–1572) introduced some reform, and less than a generation ago English teachers at last agreed that the use of it in Euclid's form was inadvisable for beginners. To-day a wider basis of assumption, including a number of intuitions, is admitted as a foundation, but the method remains the same.

Postulates of Construction.—The postulates made certain constructions permissible and restrictive. Drawing circles of any centre and radius, joining two points by a straight line, producing a straight line to any distance were the permissible, and the only permissible, ones.

Mascheroni (A.D. 1750–1800), however, in his work Geometria del Compasse, showed that a narrower foundation of construction suffices, that straight lines are unnecessary, and that the whole

of Euclid's geometry can be made dependent on the use of circles only.

Thus a straight line XY can be bisected (i.e. the mid-point lying on a straight line between X and Y can be found) thus :

With X and Y as centres and XY and YX as radii, draw circles DYE and DXE, meeting in D and E.

With E as centre and EX as radius describe an arc cutting the circle DXE in F.

With F as centre and FE as radius draw an arc cutting the circle DXE in O. Then X, Y, O are collinear.

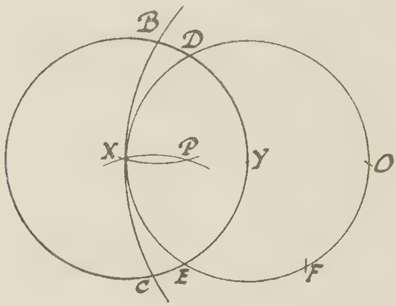


FIG. 1

With O as centre and OX as radius describe a circle cutting the circle DYE in B and C.

With B and C as centres and BX and CX as radii describe arcs cutting in P. Then P is the mid-point of a straight line XY.

By symmetry X, P, Y are collinear.

In $\triangle$s OBX and BXP,

OB = OX and BX = BP, $\therefore$ the $\triangle$s are isosceles

and $\angle$BXP = BXO = BPX = OBX $\therefore$ the $\triangle$s are similar,

$$\therefore \frac{XP}{XB} = \frac{XB}{OX}.$$

But $$XB = XY = \tfrac{1}{2}\,OX,$$

$$\therefore \quad XP = \tfrac{1}{2}\,XB = \tfrac{1}{2}\,XY.$$

It will be seen from such an example that, although the foundations have been simplified, construction and proof have become greatly complicated.

It may appear reasonable to suppose that if Euclid's constructions are possible with circles only, they are possible with straight lines only. This is not the case. They are possible, however, with the use of straight lines and one circle.

For a fuller treatment of this question the reader is referred to RULER AND COMPASSES : HILDA P. HUDSON.

It should be noted that permission to produce a line to any

distance postulates the infinity of space; so that without it the proof of the proposition, "That the exterior angle of a triangle is greater than either interior and opposite angle," would fail for large triangles in a finite plane.

The Parallel Axiom and Non-Euclidean Geometry.—Classified sometimes among the postulates, sometimes among the axioms, is the famous parallel axiom, *That if a straight line meet two other straight lines so as to make the interior angles on the same side together less than two right angles, then the straight lines when produced will meet at a finite distance and on the side of these angles.*

There are comparatively few propositions which are not dependent on this axiom or some substitute for it, e.g., PLAYFAIR'S AXIOM. The consideration of it has given rise to some of the most interesting developments in geometry. The simpler aspects of them will be dealt with here; for fuller treatment the reader is referred to NON-EUCLIDEAN GEOMETRY AND TRIGONOMETRY: H. S. CARSLAW.

Perhaps the idea that most intuitively underlies parallelism is that of equidistance, and it might be supposed that the geometry of parallel straight lines, similar triangles, etc., might be deduced from this idea. Thus, if XY is a straight line and A, B, C are three points coplanar with it, such that the perpendiculars from

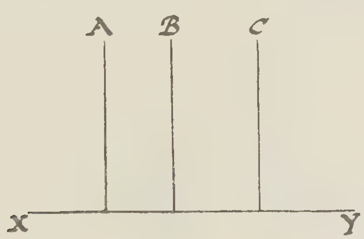


FIG. 2

A, B, and C to it are equal, it would seem to present no great difficulty to prove that A, B, and C are collinear. But it cannot be done.

Again, if any of the following propositions could be proved independently of the properties of parallel straight lines, we could dispense with this axiom which, it is to be noted, is, like Playfair's axiom, an assumption: *

(1) There is such a figure as a rectangle.
(2) The angle-sum of a triangle is two right angles.
(3) There are similar triangles of different sizes.
(4) In a plane there can be drawn through a given point only

* In some of these cases, Euclid's geometry is only established if the Postulate of Archimedes is also assumed; in this connexion it is equivalent to assuming that a straight line is infinite.

one line which does not meet a given straight line (PLAYFAIR'S AXIOM). And many others.

Mathematicians of all ages have felt that the parallel axiom should be susceptible of being proved (*see* FRANKLAND'S THEORIES OF PARALLELS: CAMBRIDGE PRESS); and at last GIROLAMO SACCHERI (1667–1733), in 1733, published *Euclides ab omni nævo vindicatus*, i.e., "Euclid cleared from every blot," in which it is postulated that the parallel-axiom is not necessarily true. His

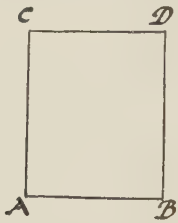


FIG. 3

fundamental assumption was that if AC and BD are perpendicular to AB and of equal length, CD may be greater or less than AB; whereas, the assumption of the parallel-axiom leads to CD = AB.

It can be proved that

(1) The angle-sum of ACDB $<$, $=$, or $>$ 4 right angles, according as CD $>$, $=$, or $<$ AB.

(2) The angle-sum of a triangle $<$, $=$, or $>$ 2 right angles, according as CD $>$, $=$, or $<$ AB.

(3) If the difference between the angle-sum of a triangle and 2 right angles is e, then the area of the $\triangle \propto e$; and hence

(4) there are no similar $\triangle$s; and so on.

But he concluded with a *volte-face* by asserting that the two alternative hypotheses were untrue.

It was, however, the first attempt to deal with the question by the method of *reductio ad absurdum*. If he expected to justify Euclid by arriving at an absurdity (i.e. by showing that some of his deductions from any one assumption were inconsistent) he was disappointed.

Other leading mathematicians turned their attention to this problem. GAUSS (1777–1855) is known to have made a searching investigation of the possibility and properties of a geometry in which a parallel postulate, different from Euclid's, and contradicting it, should be assumed.

Two younger men of his generation established the possibility of and developed a system of geometry in which the angle-sum of a triangle is less than two right angles—called the **Hyperbolic** geometry. JOHN BOLYAI (1802–1860) had succeeded in this in 1823, at the age of twenty-one, and he published his results as an appendix to a work of his father's in 1832. The discovery of this system is usually

associated with the name of LOBACHEWSKI (1793–1856), who is known to have read a paper on it in Kasan in 1826. The memoir he published in 1829–1830 *On the Principles of Geometry* was an extract from his paper of 1826, and contains a satisfactory account of the hyperbolic geometry. But he continued to publish works on this subject, of which the best known is *Geometrical Studies in the Theory of Parallels* (1840).

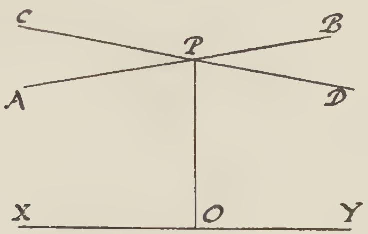


FIG. 4

In this geometry if PO is perpendicular to XY, there are two straight lines through P, APB and CPD, such that all straight lines through P within the angle APD intersect XY, but that AB and CD do not meet XY.

AB and CD are called **parallels,** and the ∠ OPA the **angle of parallelism** for the distance PO.

As PO increases, ∠ OPA diminishes.

In the Euclidean or **parabolic** geometry AB and CD coincide, and the angle OPA is constant, a right angle.

Even then the third hypothesis was regarded as untenable, till BERNHARD RIEMANN (1826–1866), Professor at Göttingen, did for it what Lobachewski did for the first. His inaugural essay, when he assumed the professorial chair in 1854, ON THE HYPOTHESES WHICH LIE AT THE BASES OF GEOMETRY, dealt concisely with this hypothesis, known as the **elliptic** geometry ; in this

(i) There is no straight line which does not meet every other straight line in its plane.

(ii) The angle-sum of a triangle is greater than two right angles.

(iii) All straight lines are closed (i.e. return to themselves), are of finite length, and, since two straight lines can coincide, of the same length.

(iv) All perpendiculars to a straight line meet in the same point, called the pole.

It will be seen that straight lines on a plane in the elliptic geometry are analogous to great circles on the surface of a sphere in Euclidean spherical geometry.

It is a curious fact that in the elliptic geometry it is not necessarily true that the exterior angle of a triangle is greater than

either interior and opposite angle, any more than it is in a Euclidean spherical triangle.

For, taking the usual notation, if BE is longer than half of the length of the whole straight line, F is not outside the triangle, but somewhere within, as at F_1, and the proof fails.

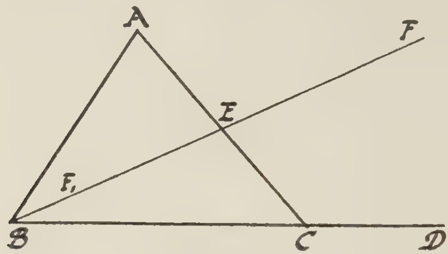


FIG. 5

Lobachewski and Riemann did not prove that these geometries were necessarily true. They put them into the same position as Euclid's, i.e. they made them dependent on an original assumption. If the assumption were true, then the geometry was logically proved; startling as the assumptions might appear to be, it could not be proved that they are false.

To make geometry independent of any special parallel-assumption is to revolutionize mathematical methods; for example, if similar triangles cannot be used, the usual definitions of sine and cosine must be abandoned; they may be replaced by the series

$$\sin \theta = \theta - \theta^3/3! + \theta^5/5! - \ldots$$
$$\cos \theta = 1 - \theta^2/2! + \theta^4/4! - \ldots$$

In the same way Newton's law of gravitation must be restated, the form in which it was given being dependent on the Euclidean assumption.

The Euclidean geometry owes its early development and recognized position to its greater simplicity; it cannot claim any greater degree of probability than the others. Practical measurement, which is inevitably approximate, would not serve to decide between the different hypotheses. Indeed, it is proved (*see* "Non-Euclidean Geometry" by Carslaw) that if the Euclidean geometry is true, the others are true also.

Three Classic Problems.—Of the three classic Greek problems two will be dealt with in this chapter—the duplication of the cube and the trisection of an angle; to the third, the squaring of the circle, a special chapter must be allotted.

To square the circle is to reduce by Euclidean methods the circle to a square of equal area. Any rectilinear figure can be reduced to a square of equal area. To reduce the area of a curvilinear figure to a square proved a very different problem.

To duplicate the cube is to find by Euclidean methods the edge of a cube whose volume is double the volume of a given cube. This problem has been credited with this origin—whence it is known as the Delian problem:—In 430 B.C. the Athenians, suffering from a plague, consulted the oracle of Apollo at Delos. Apollo replied that the plague would cease if they doubled the size of his altar, which was a cube. They at once made an altar of double the edge, and the indignant god increased the violence of the plague. The problem was then referred to the mathematicians. It is, of course, equivalent to finding a geometrical construction for a line of length $\sqrt[3]{2}$ of a unit-length. Greek geometry could construct $\sqrt{2}$, and $\sqrt{n}$; this achieved, it would be natural to attempt to construct $\sqrt[3]{2}$ and $\sqrt[3]{n}$; just as later, when mathematicians had succeeded in solving a quadratic equation, they set to work on the cubic. Similarly, when they had succeeded in bisecting a straight line or angle, they would try to trisect it or to divide it into any integral number of parts. The division of the straight line was obtained by the use of similar triangles; the general division of an angle, which would make possible the construction of any regular polygon, was not obtained. Nor was the construction of $\sqrt[3]{2}$.

The squaring of the circle has probably stimulated more mathematical activity, and has proved of more essential importance than any other problem, and will be dealt with separately. It must be remembered that for the others, as for the squaring of the circle, it was an essential condition that they should be solved, using only the straight line and circle postulates of construction.

Practically the circle could be squared if it were permitted to roll a circular disc along a straight line and measure the distance traversed in one revolution.

Duplicating the Cube and Trisecting an Angle.—Practically the other two problems could be solved if the methods of "insertions" and linkages were permitted.

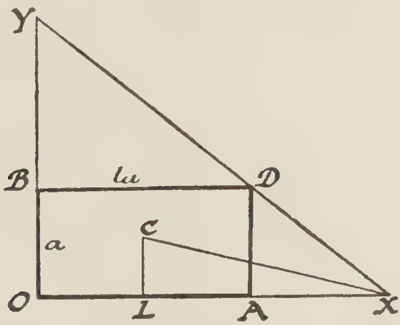


FIG 6

Thus APOLLONIUS (260–200 B.C.) gave the following method of finding $\sqrt[3]{l}$.

Let OADB be a rectangle having OB $= a$, OA $= la$, and let C be the point of intersection of OD and AB.

Then, if through D a straight line can be drawn, meeting OA and OB produced in X and Y, and such that CY = CX, then AX is $a\sqrt[3]{l}$ and BY is $a\sqrt[3]{l^2}$.

For, draw CL perpendicular to OA

Then $$CX^2 = CA^2 + AX^2 + 2LA \cdot AX$$
$$= CA^2 + AX^2 + OA \cdot AX$$
$$= CA^2 + OX \cdot AX;$$

and similarly $$CY^2 = CB^2 + OY \cdot BY;$$

i.e. $$OX \cdot AX = OY \cdot BY \quad \text{(I)}$$

Now let $$\frac{BD}{BY} = \frac{AX}{AD} = r.$$

Then $$BY = la / r \text{ and } AX = ar;$$

and substituting in (I) we get

$$(la + ar) \cdot ar = (a + la / r) \cdot la / r;$$

i.e. $$(la + ar)\, ar^3 = (ar + la)\, la;$$

$$\therefore \quad r^3 = l,$$

$$r = \sqrt[3]{l};$$

i.e. if OA is twice OB, l is 2 and AX is OB $\times \sqrt[3]{2}$.

The construction of the line XDY is possible by a mechanical means devised by Apollonius.

The first method for trisecting an angle was obtained by Hippias of Elis about 420 B.C. He invented a curve known as the Quadratrix and devised an instrument for constructing it.

It can be plotted thus :

Let AOB be a quadrant of a circle. By continued bisection AO can be divided into 2^n and $\angle$ AOB into the same number of equal parts.

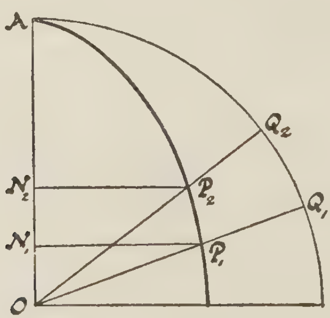


Fig. 7

Let ON_1, N_1N_2, etc., be the parts of the line; BOQ_1, Q_1OQ_2, etc., the parts of the angle. Lines through N_1, N_2, etc., parallel to

OB, meet OQ_1, OQ_2, etc., in P_1, P_2, etc. Then the smooth curve drawn through the points P_1, P_2, etc., is the **Quadratrix.**

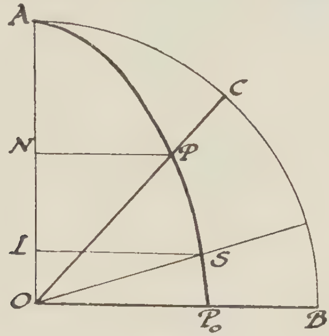


FIG. 8

To trisect an angle COB. Let OC meet the quadratrix in P. Draw PN parallel to OB. Take L in ON so that $OL=\frac{1}{3}ON$. Draw LS parallel to OB, meeting the quadratrix in S. Then $\angle SOB=\frac{1}{3}\angle COB$. For, by the law of the quadratrix,

$$\angle SOB : \text{a right } \angle = OL : OA,$$
$$\angle POB : \text{a right } \angle = ON : OA;$$
$$\therefore \angle SOB : \angle POB = OL : ON = 1 : 3.$$

In the same way any angle can be divided into any number of equal parts, and the quadratrix could be used for the practical construction of any regular polygon.

The name Quadratrix is given to the curve because it can be used to square a circle; for if the quadratrix meet OB in P_0, then OP_0 can be proved to be $\frac{2OB}{\pi}$; i.e. the ratio $\pi : 1$ can be obtained from OB and $\frac{1}{2} OP_0$.

ARCHIMEDES (287–212 B.C.) gave the following method for trisecting an angle:

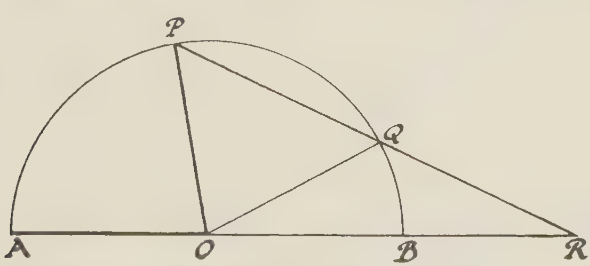


FIG. 9

Let AOP be the angle to be trisected. With O as centre,

describe a semicircle, and let AB be a diameter. Through P draw a line PQR, meeting the circumference in Q and AB produced in R, and such that QR = OP.

Then $\angle QOR = \angle QRO,$

and $\angle OPQ = \angle PQO$
$= \angle QOR + \angle QRO$
$= 2 \angle QRO;$

and $\angle AOP = \angle OPR + \angle PRO = 3 \angle PRO,$

i.e. $\angle PRO$ is $\frac{1}{3}$ of the given angle.

POQR, a linkage of three equal rods, is called the **Trisectrix linkage.**

Another form of the method, quoted by PAPPUS (third century A.D.), and possibly due to him, is as follows:

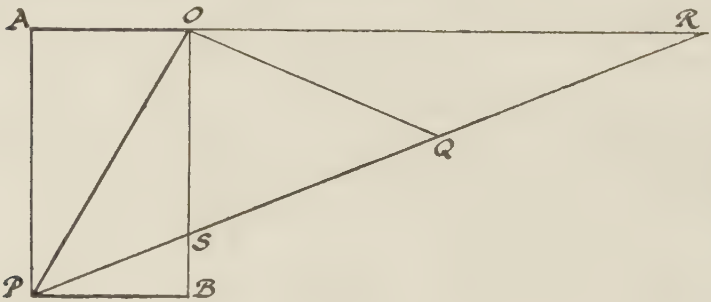


FIG. 10

Let AOP be the given angle. Complete the rectangle APBO. Draw the line PSR so that the part SR inserted between OB and AO produced is twice PO. Bisect SR in Q.

Then POQR is the figure of the trisectrix linkage; it can be constructed by the use of the conchoid (*see* p. 83).

The linkage can be made and used in the following way:

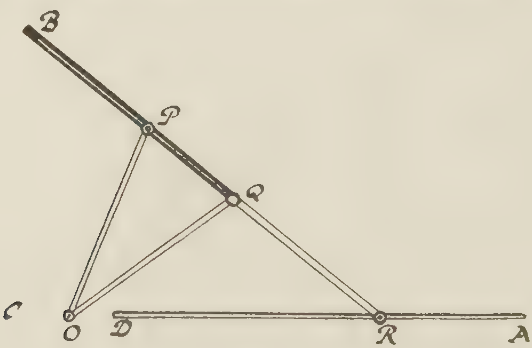


FIG. 11

In a plane table along the straight line COA cut a straight groove DA. Take two rods OP and OQ, of equal length, pivoted at the fixed point O. Take a third rod RQPB, having a peg at R to slide along DA. Pivot OQ with BR at Q, so that QR = OQ. Along QB cut a groove along which a peg inserted at P in OP may slide.

In every position $\angle$ QRO = $\frac{1}{3}$ $\angle$ COP. (*See* also p. 36.)

These problems could not be solved by straight-line and circle constructions; but it has remained for modern mathematicians using modern methods to prove it. As long as the impossibility had not been demonstrated research continued and, although failing to solve the problems themselves, resulted in the discovery of many curves and their properties.

ARCHYTAS, about 400 B.C., recognizing no restrictions of construction, obtained the first known solution of the problem of duplicating the cube. It may be reduced to the following:

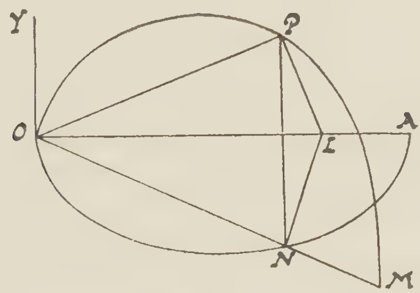


FIG. 12

ONA is a semicircle of radius a, OPM is an equal semicircle in a plane $\perp$ to ONA, and such that PN is $\perp$ to OM and $\angle$ POA is 60°.

Let OP be r and ON be x, and draw PL $\perp$ to OA, then NL is also $\perp$ to OA and $\because$ $\angle$ POA = 60°, $\therefore$ OL = $\frac{1}{2}$ OP.

Using a well-known geometrical property of the chord of a semicircle we have

$$OP^2 = ON \cdot OM, \text{ i.e. } r^2 = 2a \cdot x$$

and $$ON^2 = OL \cdot OA, \text{ i.e. } x^2 = \frac{r}{2} \; 2a.$$

Eliminating r, we get $$x^3 = 2a^3,$$

i.e. $$x = \sqrt[3]{2} \cdot a.$$

Archytas' construction gave P as the intersection of three surfaces: (I) The right circular cylinder, of which ONA is half the base and of which OY is a generator; (ii) the surface traced

out by the revolution of the semicircular arc OPM about an axis OY; (iii) a cone of semi-vertical angle 60° having an axis OA.

To PLATO is attributed the following:

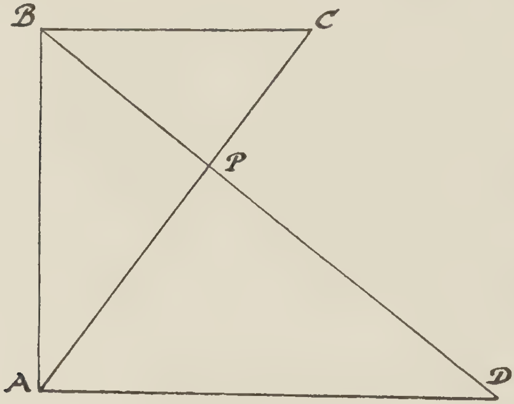


FIG. 13

Quadrilaterals ABCD can be constructed so that angles ABC, BAD are right angles, and BD and AC intersect at right angles at P; if the special condition that PD = 2 PC can be satisfied, then $BP = PC \times \sqrt[3]{2}$.

For if $\quad BP = k \,.\, PC$,

then by similar triangles, $\quad AP = k \,.\, BP = k^2PC$,

$$PD = k \,.\, AP = k^3PC;$$

$$\therefore \quad k^3 = 2.$$

MENAECHMUS (375–325 B.C.), the first mathematician to discuss conic sections, obtained $\sqrt[3]{2}$ by the intersection of conics. In modern notation the proof of his methods would appear:—

(i) the parabolas $y^2 = 2ax$ and $x^2 = ay$ intersect where $x^3 = 2a^3$. It will be noted that these equations are the same as those we obtained in Archytas' method.

(ii) the parabola $x^2 = ay$ and the hyperbola $xy = 2a^2$ also intersect where $x^3 = 2a^3$.

These conics can be constructed mechanically or plotted, i.e. an approximate or practical solution can be obtained.

NICOMEDES (second century B.C.) invented the **conchoid** (Gk. κόγχος, a *mussel* or *cockle*), the *shell-shaped* curve. It can be plotted thus (*see* Fig. 14):

Take BO a fixed straight line and PB ⊥ to it. Join P (called the "pole") to any point S in BO and produce PS to R, so that SR is a given length. Then R is on the **conchoid**. (For the mechanical construction *see* p. 81.)

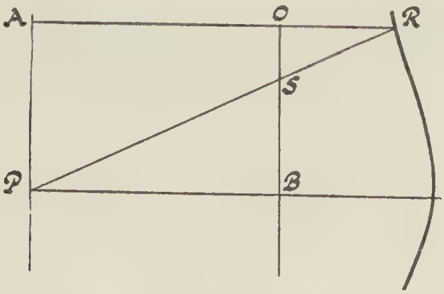


FIG. 14

To use the conchoid for the trisection of an angle, let OPB be the angle. Complete the rectangle PBOA; then taking P as the pole and OB as the fixed line, plot on the side remote from P a conchoid such that the constant distance = 2PO. (*See* Fig. 15).

Let AO meet the conchoid in R and PR meet OB in S. Then SR = 2PO.

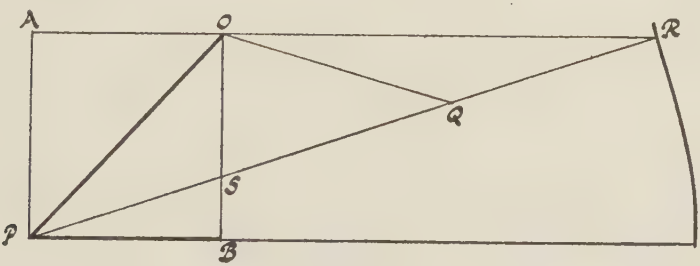


FIG. 15

Bisect SR in Q. Then POQR is the trisectrix linkage and $\angle$ ORQ = $\frac{1}{3}$ $\angle$ AOP = $\frac{1}{3}$ $\angle$ OPB. (*See* p. 36.)

To use it for the duplication of the cube, let ALGB be a rectangle. AL = $2a$, AB = $2al$. Produce GB to E so that BE = AL. Bisect BG in Q and erect QZ $\perp$ to BG, so that GZ = al. (*See* Fig. 16.)

Draw the conchoid, with Z as pole, a line through G parallel to EZ as the fixed line and with the given distance equal to ZG. Let the conchoid cut EG in K. Join KL and produce to meet BA in M.

Let AM be x and GK be y.

Then, in the triangles AML and LGK,

$$\frac{x}{2a} = \frac{2al}{y}, \text{ i.e. } x = \frac{4a^2l}{y}.$$

In $\triangle$ EZK, GN is parallel to EZ;

$$\therefore \quad \frac{\text{ZN}}{4a} = \frac{al}{y}, \quad \text{ZN} = \frac{4a^2l}{y} = x.$$

Now in $\triangle$s QZG and QZK,

$$QZ^2 = (al)^2 - a^2 = (x + al)^2 - (a + y)^2,$$

$$\therefore (a + y)^2 - a^2 = (x + al)^2 - (al)^2,$$

or $$(2a + y)\,y = (x + 2al) \,.\, x,$$

i.e. $$\frac{x + 2al}{2a + y} = \frac{y}{x} \quad \ldots\ldots\ldots\ldots \quad (\mathrm{I})$$

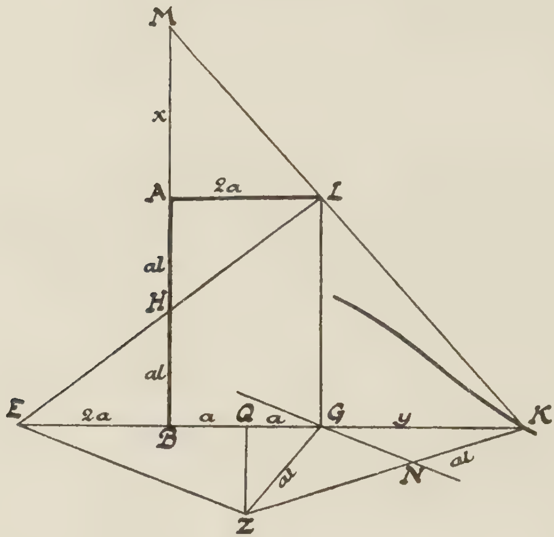


FIG. 16,

But in $\triangle$s BKM, AML, and LGK

$$\frac{x + 2al}{2\,a + y} = \frac{x}{2a} = \frac{2al}{y};$$

$$\therefore \text{ using (I)} \quad \frac{y}{x} = \frac{x}{2a} = \frac{2al}{y}.$$

Whence $$\frac{y}{x} \cdot \frac{2al}{y} = \left(\frac{x}{2a}\right)^2,$$

and $$x^3 = 8a^3 l,$$

or $$x = \mathrm{AL}\sqrt[3]{l};$$

and if l is 2, the cube of edge AL is duplicated.

DIOCLES (second century B.C.) invented the **cissoid** (Gk. *κισσός*, *ivy*), the ivy-leaf curve. It can be plotted thus (*see* Fig. 17):

Take OA the diameter of a circle, CD a diameter at right angles to it.

Let OQ be any chord through O meeting the circumference in Q and CD in S. From SO cut off SP equal to SQ.

P is a point on the cissoid.

Take OA and a line through O perpendicular to OA as axes. Let the radius of the circle be a and let P be (x,y).

Draw PM and QN perpendicular to OA.

Then $$AN = OM = x,$$

and $$QN^2 = ON \cdot NA = (2a - x)x;$$

also $$QN : NA = OM : PM,$$

i.e. $$QN = \frac{x^2}{y};$$

$$\therefore \frac{x^4}{y^2} = (2a - x)x,$$

i.e. $$x^3 = (2a - x)y^2.$$

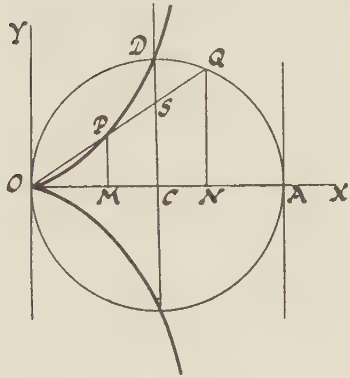


Fig. 17

(Diocles' construction was to take MC = CN and draw perpendiculars to OA at M and N, the latter meeting the circumference at Q; the line OQ meets the perpendicular at M in P. *See* also pp. 79–81).

To duplicate the cube produce CD to E so that CE = $2a$.

Let AE meet the cissoid in P.

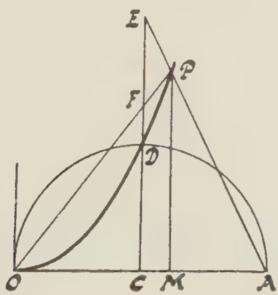


Fig. 18

Then $$PM : MA = EC : CA,$$

i.e. $$PM = 2MA,$$

i.e. $$y = 2(2a - x).$$

Substituting in $$x^3 = (2a - x)y^2,$$

we get $$2x^3 = y^3;$$
i.e. $$PM = OM \times \sqrt[3]{2}.$$

And if OP meets CE in F,
$$CF : OC = PM : OM$$
$$CF = a\sqrt[3]{2}.$$

N.B.—If CE is made $n \cdot a$ in length, the corresponding $CF = a\sqrt[3]{n}$.

It was to such constructions that Plato took exception. The greatest Greek philosopher of his day, his influence was sufficient to settle the abstract character of Greek geometry and to limit construction to those possible by straight line and circle. François Viète (1540–1603) was the first to show that constructions thus prescribed cannot solve either of the two problems.

Proofs using the methods of Cartesian geometry are here given:

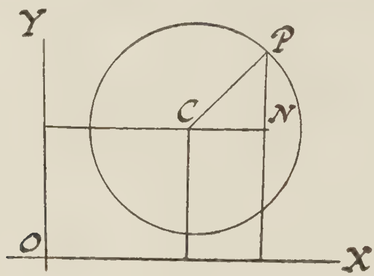


Fig. 19

Let C the centre of a circle of radius r be (d, e).

Let P a point on the circumference be (x, y).

Then drawing perpendiculars to OY and OX through C and P, and letting N be the point of intersection we have
$$CP^2 = CN^2 + PN^2;$$
i.e.
$$r^2 = (x \sim d)^2 + (y \sim e)^2$$
$$= x^2 + y^2 - 2dx - 2ye + d^2 + e^2.$$

This may be written
$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots\ldots\ldots\ldots \quad \text{(I)}$$
where there is no xy term and the coefficients of x^2 and y^2 are unity.

Now the equation of any straight line being of the 1st degree can be written
$$y = ax + b \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(II)}$$

Where this meets the circle the values of x and y for the points of intersection are obtained by eliminating y or x from (I) and (II).

In each case a quadratic equation is obtained; i.e. square roots but not cube roots may appear in the solution.

Therefore the intersection of a straight line and a circle will not solve the problem of duplicating the cube. Now consider the intersection of two circles
$$x^2 + y^2 + 2gx + 2fy + c = 0$$
and
$$x^2 + y^2 + 2lx + 2my + n = 0.$$

Subtracting, we get $2(g-l)x+2(f-m)y+c-n=0$, an equation of the 1st degree and therefore of a straight line; i.e. the points of intersection of two circles lie on a straight line and can be obtained by the intersection of one of the circles and the straight line. And therefore the intersection of circles will not solve the problem of duplicating the cube.

To find θ so that it shall be one-third of a given angle A we must solve some such equation as

$$\sin 3\theta = \sin A\ ;\ \cos 3\theta = \cos A\ ;\ \tan 3\theta = \tan A.$$

But

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta,$$

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$$

$$\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2\theta};$$

i.e. each equation reduces to a cubic, which, as we have just seen, cannot be solved by the intersection of straight lines and circles. And so the trisection of an angle cannot be solved by line and circle constructions.

The examples, given in this chapter to show the limitations of the Euclidean constructions, serve also to show the inventive powers and the grasp of geometry possessed by the Greek mathematicians during the two or three centuries of Greek mathematical discovery.

CONSTRUCTION OF REGULAR POLYGONS.

The last point we shall touch on is the possibility of constructing a regular polygon of n sides.

The general consideration of the possibility will be found to be associated with the Theory of Numbers. Thus Euclid having described a regular 3-gon and a regular 5-gon, was able to construct a regular 15-gon, because 15 is the L.C.M. of the primes 3 and 5.

The method depends on these considerations:—If $n = pq$, where p and q are prime to each other, and if α is the angle at the centre of a circle subtended by one side of the regular n-gon, then $p\alpha$ is the angle subtended by one side of the regular q-gon and $q\alpha$ is the angle subtended by the regular p-gon.

Now by the Long Division process for H.C.F., integers l and m may be found, less respectively than p and q, such that $lq \sim mp = 1$. Therefore $lq\alpha \sim mp\alpha = \alpha$. If, then, the p-gon and q-gon can be described, the n-gon in question can be obtained as follows: In a circle describe the p-gon and q-gon having a common angular point A. Let AL be the arc for l sides of the p-gon and AM the arc for m sides of the q-gon; then the chord LM is one side of the n-gon. For example, suppose a 7-gon and an 11-gon could be constructed.

$$
\begin{array}{c|c|c|c}
1 & 7 & 11 & 1 \\
 & 4 & 7 & \\
\hline
3 & 3 & 4 & 1 \\
 & 3 & 3 & \\
\hline
 & & 1 &
\end{array}
$$

The H.C.F. process shows that

$$\frac{7}{11} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}}}.$$

Now

$$\cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} = \frac{2}{3}.$$

2 and 3 are the values of l and m corresponding to the values 7 and 11 for p and q.

We have $$2 \times 11 - 3 \times 7 = 1;$$

i.e. if AL is the arc of two sides of a 7-gon and AM the arc of three sides of an 11-gon inscribed in the same circle, LM is the arc of one side of a 77-gon.

These considerations do not cover the cases where p and q have a common factor. And they leave us with the problem to solve of constructing an n-gon where n is prime. The latter problem is associated with the problem of dividing an angle (or the particular angle 360°) into n equal parts.

The 7-gon and the 11-gon mentioned above cannot be constructed. Indeed, the only prime values of n below 10,000 for which the n-gon can be constructed are 3, 5, 17, 257. For a fuller discussion of the question and the actual construction of the 17-gon, the reader is referred to Ruler and Compasses by Hilda P. Hudson in this series.

Approximate Constructions.—Where an exact construction is impossible, an approximate one may be of interest. For the **trisection of an angle,** the following is one of many (*see* Fig. 20):

Let AOB be the angle; take OA and OB equal. With centres A and B and radii AB and BA draw arcs to intersect in C; to cut OB in F and G and OA in D (as in the Fig.). Join DG and CF by straight lines intersecting in X. Draw XL perpendicular to OB.

The angle XOL is approximately $\frac{1}{3}$ AOB.

An outline of the investigation is given.

Let $\angle$ AOB $= 2\alpha$, then $\angle$ OCB $= 30°$, CBG $= 30° + \alpha$, BFC $=$ FCB $= 15° + \dfrac{\alpha}{2} = \beta$ (say); and let $\angle$ DGO $= \varphi$.

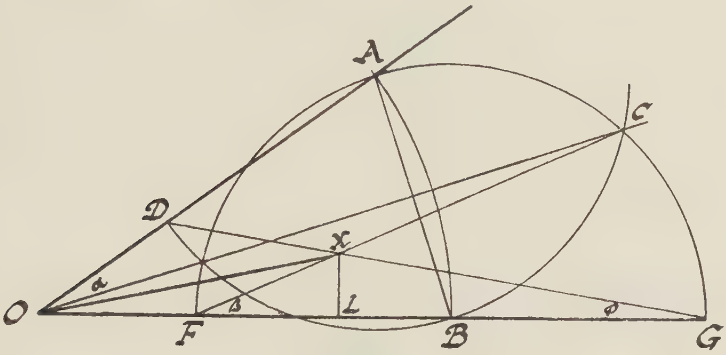


Fig. 20

Let OA $= a$.
Then in $\triangle$ DOG,

$$\frac{\tan \frac{1}{2}(\mathrm{ODG} - \mathrm{OGD})}{\tan \frac{1}{2}(\mathrm{ODG} + \mathrm{OGD})} = \frac{\mathrm{OG} - \mathrm{OD}}{\mathrm{OG} + \mathrm{OD}} = \frac{2\mathrm{FB}}{2\mathrm{OB}}$$

$$\frac{\tan(90° - \varphi - \alpha)}{\tan(90° - \alpha)} = \frac{4 \sin\alpha}{2};$$

i.e.
$$\cot(\varphi + \alpha) = 2\cos\alpha \quad \ldots\ldots\ldots\ldots \quad \text{(I)}$$

In $\triangle$ FXG,

$$\mathrm{XG} = \frac{2b \sin\beta}{\sin(\varphi + \beta)};$$

where
$$b = \mathrm{FB} = \mathrm{BC} = \mathrm{AB} = 2a \sin\alpha \ldots\ldots\ldots \quad (1)$$

$$\cot \mathrm{XOL} = \frac{\mathrm{OG} - \mathrm{LG}}{\mathrm{LX}} = \frac{(a + b) - \mathrm{XG}\cos\varphi}{\mathrm{XG}\sin\varphi}$$

$$= \frac{a + b}{\mathrm{XG}\sin\varphi} - \cot\varphi$$

$$= \frac{(a + b)\sin(\beta + \varphi)}{2b\sin\beta\sin\varphi} - \cot\varphi, \text{ using (1)}$$

$$= \frac{a + b}{2b}(\cot\beta + \cot\varphi) - \cot\varphi$$

$$= \frac{a}{2b}(\cot\beta + \cot\varphi) - \tfrac{1}{2}(\cot\varphi - \cot\beta)$$

$$= \frac{(\cot\beta + \cot\varphi)\operatorname{cosec}\alpha}{4} - \tfrac{1}{2}(\cot\varphi - \cot\beta) \ldots \text{(II)}$$

φ is determined approximately from tables by (I) and substituted in (II), which gives $\angle$ XOL. The construction gives 13° 16′, 10° 1′ and 6° 45′ for $\angle$ XOL, when $\angle$ AOB is 40°, 30° and 20°.

Dr. Benchara Branford has kindly given permission to introduce the following construction which appears in his AIMS OF A MATHEMATICAL EDUCATION.

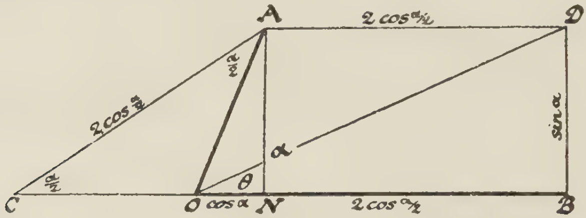


FIG. 21

Let AOB be the angle. Take OA a unit length; produce BO to C so that OC = OA.

Draw AD parallel to OB so that AD = AC.

Then $\angle$ DOB = nearly $\frac{1}{3}$ $\angle$ AOB.

Draw AN and DB perpendicular to OB.

Then if $\angle$ AOB is α and $\angle$ DOB is θ,

$$AD = AC = 2\cos\frac{\alpha}{2},$$

$$DB = \sin\alpha,\quad OB = 2\cos\frac{\alpha}{2} + \cos\alpha,$$

and
$$\cot\theta = \frac{2\cos\frac{\alpha}{2} + \cos\alpha}{\sin\alpha} = \operatorname{cosec}\frac{\alpha}{2} + \cot\alpha.$$

Construction and analysis are extremely simple; the investigation of the degree of accuracy and range of applicability is left to the reader.

To Construct a Regular *n*-gon.—The following is a simple and well-known construction (*see* Fig. 22):

Let AB, the diameter of a semicircle APB, be divided into n parts, each of length d; let AQ = $2d$.

On AB describe the equilateral triangle ACB.

Join CQ and produce to meet the semicircle in P.

Then AP is approximately one side of the regular n-gon inscribable in the complete circle.

$$OP = \frac{nd}{2},\quad CO = \frac{nd\sqrt{3}}{2},\quad QO = \frac{n-4}{2}\cdot d$$

Let $\angle PCO = \alpha$, and $\angle CPO = \beta$.

Then,
$$\sin\alpha = \frac{n-4}{2\sqrt{n^2-2n+4}},$$

$$\cos\alpha = \frac{n\sqrt{3}}{2\sqrt{n^2-2n+4}}.$$

And since in $\triangle$ COP

$$\sin\beta = \sqrt{3}\sin\alpha,$$

$$\sin\beta = \frac{(n-4)\sqrt{3}}{2\sqrt{n^2-2n+4}},$$

$$\cos\beta = \frac{\sqrt{n^2+16n-32}}{2\sqrt{n^2-2n+4}}.$$

Whence $\theta = 90^\circ - (\alpha + \beta)$

$$= 90^\circ - \tan^{-1}\frac{n-4}{n\sqrt{3}} - \tan^{-1}\frac{\sqrt{3}\,(n-4)}{\sqrt{\{(n+8)^2-96\}}}$$

$$= 90^\circ - \cot^{-1}\frac{n\sqrt{3}}{n-4} - \cot^{-1}\frac{\sqrt{\{(n+8)^2-96\}}}{\sqrt{3}\,(n-4)}.$$

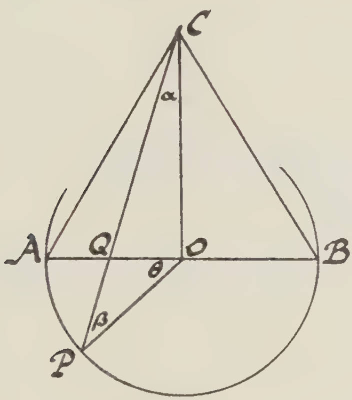


FIG. 22

Tables can be used to give θ approximately, or we can proceed thus:

$$\sin\theta = \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \frac{\sqrt{3}\,\{\,n\sqrt{(n+8)^2-96} - (n-4)^2\,\}}{4\{(n-1^2)+3\}} \quad \ldots\ldots \text{(I)}$$

This formula shows AP to be correct for regular figures of 3, 4 or 6 sides and to be within 1 per cent of correctness for n-gons where n is not greater than 10.

But the reader is advised to complete the investigations of both constructions for himself and to test them in a number of cases.

An approximate construction for a regular pentagon is given on p. 16 (*see* Dürer).

CHAPTER IV

SQUARING THE CIRCLE

$\pi = 3{\cdot}14159$	26535	89793	23846	26433
83279	50288	41971	69399	37510
58209	74944	59230	78164	06286
20899	86280	34825	34211	70679
82148	08651	32823	06647	09384
46095	50582	23172	53594	08128

etc.

Squaring the Circle—The Nature of the Problem.—The Greek geometer was able to reduce any rectilinear figure to a square of equal area with the use of circle and straight-line constructions. He would naturally seek to extend his processes to curvilinear areas, and in particular to reduce the area of a circle to that of a square of equal area.

The beginner in geometry can convince himself by measuring circumferences and diameters of circular objects, by counting the squares in a circle drawn on squared paper, by weighing a circular sheet of metal and comparing its weight with that of a measurable area of the same metal of the same thickness, that the circumference of a circle is $2\pi r$ and the area is πr^2, where r is the radius and π some number whose value is about $3{\cdot}14$. Greek geometry suffices to prove the formulæ $2\pi r$ and πr^2, but does not succeed in evaluating the number π nor in determining the nature of the number.

This problem is the problem of the **quadrature of the circle.** It is closely allied to the problem of the **rectification of the circle,** that is, the reduction of the circumference of the circle to a straight line of equal length, also with the use of only circle and straight-line constructions. For it can be proved that the area of a circle is the area of a rectangle formed by the radius and the semi-circumference. To solve one of these problems therefore is to obtain a solution of the other, and we may regard the two problems as only one.

Leaving out the experimental method, as used by the beginner—a method which determines π sufficiently accurately for practical purposes but does not touch the real problem—two methods have been used:

(1) The method of Archimedes (287–212 B.C.), which regards the circle as the limit to which regular circumscribing and inscribed polygons converge both as regards their area and perimeter.

(2) The seventeenth-century method, which regards π as the limiting-sum of a series.

The first method is essentially geometrical, the second is essentially algebraical. But though the problem is in the first place a geometrical one, the final word in it has been left to algebra and has required more than two thousand years of mathematical progress to give it utterance.

History of the Symbol.—That π is now used as a symbol for the numerical value of the ratio of the circumference to the diameter of a circle is due to the authority of Euler.

Oughtred (1647) and Barrow a few years later used π for the circumference of the circle. John Bernouilli (1667) used c. Euler in 1734 used p, in a letter in 1736 he used c, but later he used π, and this has come into general use.

The Value of π.—As we shall see, we cannot evaluate π exactly; it has been calculated to more than 700 places, a number of which are given at the beginning of the chapter.

The following mnemonic* for 30 places may be of interest. Each word, by the number of its letters, records a digit:

Que j'aime à faire apprendre un nombre utile aux sages,
Immortel Archimède, artiste ingénieur !
Qui de ton jugement peut priser la valeur ?
Pour moi ton problème eut de pareils avantages.

Approximate Values.—If a circle be drawn with a circumscribed and inscribed square, it is clear that the area of the circle lies between them, but the area of the first is $4r^2$ and the area of the second is half of it or $2r^2$, i.e. π lies between 2 and 4, and is probably nearly 3.

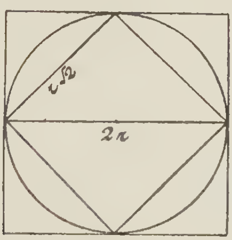


FIG. 23

The circumference is less than the perimeter of the circumscribed square and greater than that of the inscribed square, i.e. $< 8r$, but $> 4\sqrt{2}r$, i.e. $5{\cdot}656r$.

This gives $4 > \pi > 2{\cdot}828 \ldots$

The Babylonians, and in the twelfth century B.C. the Chinese,

* I have seen it quoted as being given in Dr. Kowalevski's Calculus, and I have heard it ascribed to Leverrier, the French discoverer of Neptune; but I have been unable to confirm either as fact.

used 3 as the value, and so did the Jews (*see* 1 Kings vii, 23 and 2 Chron. iv, 2). $3\frac{1}{6}$ and $3\frac{1}{8}$ were sometimes used.

Now if we could construct a square less than the circumscribing square or greater than the inscribed, placed symmetrically with respect to the circle and such that the parts of the circle outside the area common to both = the parts of the square outside this area, we should solve the problem.

AHMES in the RHIND PAPYRUS (1700 B.C.) gives this rule: Diminish the diameter of the circle by one-ninth of itself. The square on the remainder is the area of the circle.

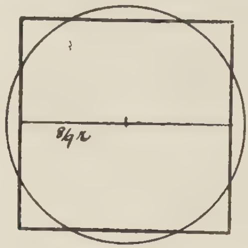


FIG. 24

This gives $\pi = 4 \times (\frac{8}{9})^2 = 3\frac{13}{81} = 3{\cdot}1605$ nearly, and this is only about 0·6 per cent too high.

ARCHIMEDES, by calculation, placed π between $3\frac{1}{7}$ and $3\frac{10}{71}$, i.e. 3·142857 . . . and 3·140845 . . . These values are less than 0·04 per cent in excess and defect. The mean of these values would be only 0·008 per cent in excess of the true value.

PTOLEMY (A.D. 87–168) gave π in sexagesimal notation as $3^\circ\ 8'\ 30''$, i.e. $3\frac{17}{120}$ or 3·1416, in which the error is less than 0·00025 per cent.

ARYA BHATA (*b.* Patna, A.D. 476) gave $3{\cdot}141\dot{6}$.

BRAHMAGUPTA (*b.* A.D. 598) gave $\sqrt{10}$. He used the same method as Archimedes, found the areas of inscribed 12-, 24-, 48-, 96-gons in a circle of unit radius to be $\sqrt{9{\cdot}65}$, $\sqrt{9{\cdot}81}$, $\sqrt{9{\cdot}86}$, $\sqrt{9{\cdot}87}$, and assumed the area of the circle to be $\sqrt{10}$ (*see* p. 238). It is in itself a very simple result, and, being 3·16228 . . ., is less than 0·7 per cent in excess.

BHASKARA (*b.* A.D. 1114), using Archimedes' method, obtained 3·1416, which he called the true value, which is less than 0·00025 per cent in excess.

ISU CH'UNG CHIH (*b.* A.D. 430) found $\frac{22}{7}$ and $\frac{355}{113}$; he deduced the latter from his results, $3{\cdot}1415927 > \pi > 3{\cdot}1415926$; $\frac{355}{113}$ is 3·1415929 . . ., and is therefore slightly greater than his higher limit, but it is a fairly simple fraction and can be written $3 + \cfrac{1}{7 + \frac{1}{16}}$.

In the sixteenth and seventeenth centuries mathematicians

using Archimedes' principle obtained more accurate results than he, and with polygons of a smaller number of sides.

Ludolf van Ceulen (1539–1610) spent a lifetime in computing π, and his achievement of obtaining 35 places was commemorated in the phrase "Ludolphian Number" applied to what we now call π.

Huygens (1629–1695), using only the triangle and hexagon, found π to be between 3·1415926533 and 3·1415926538.

Using series (either Gregory's or series derived from it)—

Sharp (1699)	obtained	72	places,	71 correct.
Machin (1706)	,,	100	,,	All ,,
Rutherford (1841)	,,	208	,,	All ,,
Rutherford (1853)	,,	440	,,	All ,,

Shanks (1873), after a contest with Richter to obtain the greatest number of places, beat Richter's 500 with 707. This is the record. He used Machin's series (p. 60).

Early Methods.—Hippias of Elis, about 420 B.C., devised a curve, the Quadratrix (*see* p. 34), which would rectify a circle; it is the first recorded method of obtaining π with theoretical correctness, but as the Quadratrix cannot be constructed by straight-line and circle constructions it does not provide a solution of the problem.

The first quadrature of a curvilinear area is that of Hippocrates (*b.* Chios, 470 B.C.). He found the area of a **lune.**

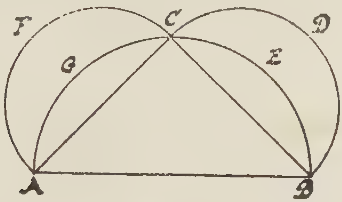


Fig. 25

Let ABC be an isosceles triangle, with $\angle$ ACB a right angle.

Let semicircles be drawn on the sides as in Fig. 25.

Then, the semicircles being similar figures drawn on the sides of a right-angled triangle, the semicircle AGCEB = the sum of the semicircles AFC and CDB.

Taking away the common areas, we have the sum of the two lunes AFCG and CDBE = $\triangle$ ACB, i.e. the lune AFCG = $\frac{1}{4}$ AC^2.

This result, by providing a case in which a curvilinear figure bounded by circular arcs had been successfully squared, would serve to encourage geometers attempting to square the circle.

But Hippocrates thought that it actually solved the problem.* His procedure, slightly modified, is as follows :—

*Sir Thomas Heath refuses to believe Hippocrates capable of such a blunder.

Let ABCD be half a regular hexagon inscribed in a semicircle on AD as diameter.

Let $AB = BC = CD = \frac{1}{2}AD = r$.

Describe semicircles outwards on AB, BC, CD.

Then the semicircle on AD + 3 lunes = the quadrilateral ABCD + the semicircles on AB, BC, and CD.

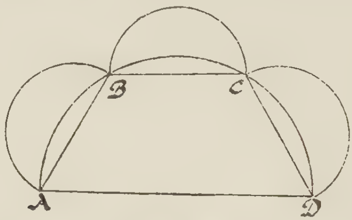


FIG. 26

And assuming that these lunes are the same as those in the case of the isosceles right angled triangles, i.e. each $= \frac{1}{4}r^2$ (this is not so),

we have $$\frac{1}{2}\pi r^2 + \frac{3}{4}r^2 = \frac{3\sqrt{3}}{4}r^2 + \frac{3}{8}\pi r^2,$$

i.e. $$\pi = 6(\sqrt{3} - 1) = 4\cdot392\ \ldots, \text{ much too high.}$$

Archimedes' Method.—Archimedes devoted much attention to the quadrature of curvilinear areas, and succeeded in squaring a segment of a parabola, but this is independent of π.

His approximation of π was obtained by a method of rectification:

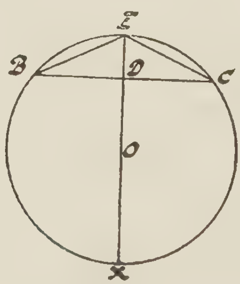


FIG. 27

Let BC be a side of a regular n-gon inscribed in a circle of centre O.

Let EDOX be the diameter perpendicular to BC.

Then BE and EC are sides of a regular $2n$-gon.

Let BC be l, BE m, and the radius r. Then by similar triangles

$$\frac{EC}{ED} = \frac{EX}{EC},$$

i.e. $$m^2 = (r - \mathrm{OD})2r.$$

But $$\mathrm{OD} = \sqrt{r^2 - \left(\frac{l}{2}\right)^2};$$

$$\therefore \quad m^2 = 2r\left\{r - \sqrt{r^2 - \left(\frac{l}{2}\right)^2}\right\}.$$

This gives a formula of reduction for obtaining the perimeter of the $2n$-gon from that of the n-gon, the process of finding a square root being the most troublesome part of the computation, though to the Greeks this must have been formidable enough.

The simplest polygon to begin with is the hexagon, since $l = r$. $\therefore$ if m_{12}, m_{24}, etc., are sides of the regular 12-gon, 24-gon, etc.,

$$\begin{aligned} m_{12}^2 &= r^2\left[2 - \sqrt{3}\right] \\ m_{24}^2 &= r\left\{2r - \sqrt{4r^2 - m_{12}^2}\right\} \\ &= r^2\left\{2 - \sqrt{4 - (2 - \sqrt{3})}\right\} \\ &= r^2\left\{2 - \sqrt{2 + \sqrt{3}}\right\}, \text{ and so on.} \end{aligned}$$

For the circumscribed polygon let OB and OC produced meet the tangent at E in F and G (Fig. 28).

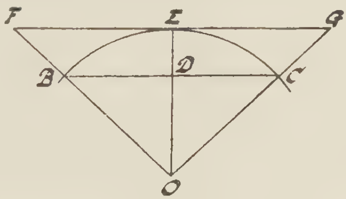


Fig. 28

Then FG is one side of a circumscribed n-gon.

Now, $$\frac{\mathrm{FG}}{\mathrm{BC}} = \frac{\mathrm{FE}}{\mathrm{BD}} = \frac{\mathrm{EO}}{\mathrm{DO}};$$

$$\therefore \quad \mathrm{FG} = l \cdot \frac{r}{\sqrt{r^2 - \left(\frac{l}{2}\right)^2}} = \frac{2lr}{\sqrt{4r^2 - l^2}};$$

i.e. the perimeter of a circumscribed polygon can be obtained from that of the corresponding inscribed polygon.

So that, if M_6, M_{12}, M_{24} are the lengths of sides of a circumscribed polygon,

$$\mathrm{M}_6 = \frac{2r^2}{\sqrt{4r^2 - r^2}}$$

$$\mathrm{M}_{12} = \frac{2rm_{12}}{\sqrt{4r^2 - m_{12}^2}}$$

and so on. And as $\sqrt{4r^2 - m_{12}^2}$ is calculated to get m_{24}, little additional working is required to get M_{12}.

The following values of perimeters may be thus obtained:

Number of Sides	Inscribed Polygon	Circumscribed Polygon
6	6·00 000	6·92 820
12	6·21 166	6·43 078
24	6·26 526	6·31 932
48	6·27 870	6·29 218
96	6·28 206	6·28 542
192	6·28 291	6·28 357
384	6·28 311	6·28 333

It is clear that when any number of digits is the same in the two columns, they will be the same in the value of 2π. Thus 96-gons give 3·14, 192-gons give 3·142 nearly, 384-gons give 3·1416 nearly for π. The mean of the values in the two columns would give still closer approximations.

These results can be graphed; and the graph shows more convincingly than the table the approach of the two graphs to the asymptotic line $y = \pi$.

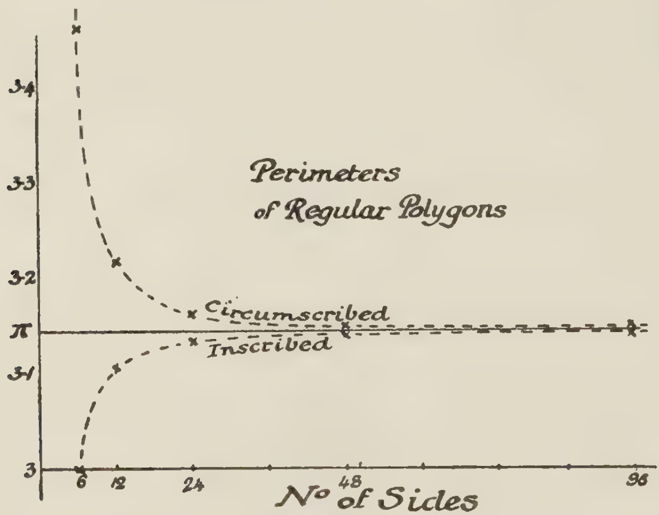


FIG. 29

This graph and a similar one for areas can be plotted by boys ignorant of formal geometry. They must inscribe in, and circumscribe about, circles of standard radius, regular polygons of various numbers of sides, and by measuring the lengths of sides

to get perimeters, or by counting squares to get areas, compile their table of values for the graph. The knowledge of a little trigonometry, however, will enable the student to obtain the values for the graph with greater accuracy and speed.

Series Methods.—VIETA (1540–1603) obtained by geometrical methods the formula

$$\frac{4}{\pi} = \left(\sqrt{\frac{1}{2}}\right) \times \left(\sqrt{\frac{1}{2} + \sqrt{\frac{1}{2}}}\right) \times \left(\sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \frac{1}{\sqrt{2}}}}\right) \times \ldots$$

to infinity.

If we write this $\frac{4}{\pi} = u_1 \times u_2 \times u_3 \times \ldots$ we see that $u_{n+1} = \sqrt{\frac{1}{2} + u_n}$.

This gives a precise statement of the value of π, but is not convenient for computation.

WALLIS (1616–1703), by an ingenious use of interpolation, obtained

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \ldots \text{ to infinity.}$$

As a statement this is simpler and more direct than Vieta's, but it involves the use of a very large number of terms in the computation to a very small degree of accuracy. If it were not for this disadvantage it would be preferable to all other series, for this reason, that at each step of the computation the result is complete and all previous working can be done away with. Thus :

3	4	p_1
	1·333333	
3	2·666667	p_2 (by subtracting $\frac{1}{3}$ of p_1 from p_1)
	·888889	
5	3·555556	p_3 (by adding $\frac{1}{3}$ of p_2 to p_2)
	·711111	
5	2·844445	p_4 (by subtracting $\frac{1}{5}$ of p_3 from p_3)
	·568889	
7	3·413334	p_5
	·487619	
	2·925715	p_6

$p_1, p_2, p_3, p_4, p_5, p_6$ converge to the value of π; $p_1, p_3, p_5 \ldots$

being too great, p_2, p_4, p_6 . . . too small. But the convergence is very slow.

Lord Brouncker (1620–1684), one of the founders of the Royal Society, gave π as an infinite continued fraction:

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 \ldots}}}}$$

This, again, is not suitable for computation, although quite definite in form.

Gregory's Series.—Gregory, in 1671, obtained the series

$$\theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \ \ldots$$

i.e.
$$\tan^{-1}\alpha = \alpha - \tfrac{1}{3}\alpha^3 + \tfrac{1}{5}\alpha^5 - \ \ldots$$

of which the particular case

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots$$

is the basis of all subsequent computations. It was first used by Sharp in 1699 at the suggestion of Halley.

It can be used thus:

$$\pi = 4(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots)$$

Positive terms	Negative terms
$4 = 4$	$\frac{4}{3} = 1{\cdot}3333$
$\frac{4}{5} = {\cdot}8$	$\frac{4}{7} = {\cdot}5714$
$\frac{4}{9} = {\cdot}4444$	$\frac{4}{11} = {\cdot}3636$
$\frac{4}{13} = {\cdot}3077$	$\frac{4}{15} = {\cdot}2667$
$\frac{4}{17} = {\cdot}2353$	$\frac{4}{19} = {\cdot}2105$
$\frac{4}{21} = {\cdot}1905$	$\frac{4}{23} = {\cdot}1739$

It will clearly require a great number of terms to get very few places accurate. The 12 terms evaluated above give $\pi = 5{\cdot}9779 - 2{\cdot}9194 = 3{\cdot}0585$; i.e. only one figure correct.

Euler (1707–1783) deduced several other series from Gregory's, the best known being:

$$\pi = 4\left(\frac{1}{2} - \frac{1}{3}\cdot\frac{1}{2^3} + \frac{1}{5}\cdot\frac{1}{2^5} - \frac{1}{7}\cdot\frac{1}{2^7} + \ldots\right)$$
$$+ 4\left(\frac{1}{3} - \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} - \frac{1}{7}\cdot\frac{1}{3^7} + \ldots\right)$$

Working as far as the 5th decimal place we have :

For the Powers of $\frac{1}{2}$			For the Powers of $\frac{1}{3}$		
2	4		3	4	
4	2·	α_1	9	1·33333	β_1
4	·5	α_3	9	·14815	β_3
4	·125	α_5	9	·01646	β_5
4	·03125	α_7	9	·00183	β_7
4	·00781	α_9	9	·00020	β_9
4	·00195	α_{11}		·00002	β_{11}
4	·00049	α_{13}			
4	·00012	α_{15}			
4	·00003	α_{17}			
	00001	α_{19}			

Introducing coefficients we get :

Positive terms		Negative terms	
α_1	2·	$\frac{1}{3}\alpha_3$	·16667
$\frac{1}{5}\alpha_5$	·025	$\frac{1}{7}\alpha_7$	·00446
$\frac{1}{9}\alpha_9$	·00087	$\frac{1}{11}\alpha_{11}$	·00018
$\frac{1}{13}\alpha_{13}$	·00004	$\frac{1}{15}\alpha_{15}$	·00001
β_1	1·33333	$\frac{1}{3}\beta_3$	·04938
$\frac{1}{5}\beta_5$	·00329	$\frac{1}{7}\beta_7$	·00026
$\frac{1}{9}\beta_9$	·00002		
	3·36255		·22096

whence $\pi = 3{\cdot}14159$

Thus using 10 terms of one part, and 6 of the other, 6 figures are obtained, all being correct.

RUTHERFORD obtained and used :

$$\pi = 16\left\{\frac{1}{5} - \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{7^3} - \dots\right\}$$

$$- 4\left\{\frac{1}{70} - \frac{1}{3}\cdot\frac{1}{70^3} + \frac{1}{5}\cdot\frac{1}{70^5} - \dots\right\}$$

$$+ 4\left\{\frac{1}{99} - \frac{1}{3}\cdot\frac{1}{99^3} + \frac{1}{5}\cdot\frac{1}{99^5} - \dots\right\}.$$

This is much more rapidly convergent and can be evaluated by Short Division.

Writing this

$$\pi = \left\{ \alpha_1 - \frac{1}{3}\alpha_3 + \frac{1}{5}\alpha_5 - \ . \ . \ . \right\} - \left\{ \beta_1 - \frac{1}{3}\beta_3 + \frac{1}{5}\beta_5 - \ . \ . \ . \right\} + \left\{ \gamma_1 - \frac{1}{3}\gamma_3 + \frac{1}{5}\gamma_5 - \ . \ . \ . \right\}$$

we can set out the computation as follows (the arrangement in numbered columns of four digits divided into pairs by spaces facilitates the final arrangement for addition and subtraction) :—

		1		2		3		4		5	
α_1	3	2									
α_3		12	8								
α_5			51	2							
α_7			2	04	8						
α_9				8	19	2					
α_{11}					32	76	8				
α_{13}					1	31	07	2			
α_{15}						5	24	28	8		
α_{17}							20	97	15	2	
α_{19}								83	88	60	8
α_{21}		The procedure is to multiply by 0.04 (i.e. to divide by 25).						3	35	54	43
α_{23}									13	42	17
α_{25}										53	68
α_{27}										2	14
α_{29}											8
β_1		05	71	42	85	71	42	85	71	42	86
β_3			8	16	32	65	30	61	22	44	90
				11	66	18	07	58	01	74	93
β_5					16	65	97	25	11	45	36
						23	79	96	07	30	65
β_7							33	99	94	39	01
								48	57	06	27
β_9		The procedure is to divide twice by 70.							69	38	66
										99	12
β_{11}										1	41
											2

		1		2		3		4		5	
γ_1		04	04	04	04	04	04	04	04	04	04
			36	73	09	45	82	18	54	91	28
			4	08	12	16	20	24	28	32	36
γ_3				37	10	19	65	47	66	21	12
				4	12	24	40	60	85	13	46
					37	47	67	32	80	46	68
					4	16	40	81	42	27	41
						37	85	52	85	66	13
γ_5						4	20	61	42	85	13

	1	2	3	4	5
			38	23 76	62 28
			4	24 86	29 14
				38 62	39 01
γ_7				4 29	15 45
				39	01 40
	The procedure is to divide twice by 99 (using the factors 11 and 9).			4	33 49
					39 41
γ_9					4 38
					40
					4

Some of the last steps of division in each partial series will be seen to be unnecessary.

Remembering the — sign before the β series and collecting positive terms we have:—

	0	1	2	3	4	5
α	3	2				
$\frac{1}{5}\alpha_5$		00 10	24			
$\frac{1}{9}\alpha_9$			91	02 22	22 22	22 22
$\frac{1}{13}\alpha_{13}$				10 08	24 61	53 85
$\frac{1}{17}\alpha_{17}$				1	23 36	18 82
$\frac{1}{21}\alpha_{21}$					15	97 83
$\frac{1}{25}\alpha_{25}$						2 14
$\frac{1}{3}\beta_3$			03 88	72 69	19 33	91 64
$\frac{1}{7}\beta_7$					06 93	86 61
γ_1		04 04	04 04	04 04	04 04	04 04
$\frac{1}{5}\gamma_5$				84	12 28	57 03
$\frac{1}{9}\gamma_9$						49
TOTAL	3	24 14	32 83	89 89	12 96	34 67

Collecting negative terms we have:

		1	2	3	4	5
$\frac{1}{3}\alpha_3$		04 26	66 66	66 66	66 66	66 67
$\frac{1}{7}\alpha_7$			29 25	71 42	85 71	42 86
$\frac{1}{11}\alpha_{11}$			2	97 89	09 09	09 09
$\frac{1}{15}\alpha_{15}$				34	95 25	33 33
$\frac{1}{19}\alpha_{19}$					4 41	50 57
$\frac{1}{23}\alpha_{23}$						58 35
$\frac{1}{27}\alpha_{27}$						8
β_1		05 71	42 85	71 42	85 71	42 86
$\frac{1}{5}\beta_5$				4 75	99 21	46 13
$\frac{1}{9}\beta_9$						11 01
$\frac{1}{3}\gamma_3$			1 37	41 46	86 95	04 49
$\frac{1}{7}\gamma_7$					61	30 78
TOTAL		09 98	40 18	53 99	33 63	96 22

and subtracting this from the total of positive terms we obtain

$$3{\cdot}14159265358979323845,$$

of which only the last figure is wrong.

This specimen of computation, in which all the working is shown, may serve to show the reader the advantage of an orderly arrangement. It will also show how the heaviness of the computation grows as the number of figures which it is desired to obtain is increased, and it will give him some concrete idea of the phrases "convergent" and "rapidly convergent" as applied to series. But he will see these things more clearly if he attempts a few evaluations of π for himself. If he can rely on working accurately even when doing mental division by 19, etc., he should be able to emulate Euler's feat of obtaining 20 places in an hour. (Euler used

$$\frac{\pi}{4} = 5\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{3}{79},$$

expanded by Gregory's series.) Only if he develops an extraordinary zest will he attempt to emulate Zacharias Dase, an expert computer, who, in 1844, obtained 200 figures in two months.

In any case, he may well ask what purpose it serves to obtain so many places, to say nothing of Shanks' 700. When we discuss the nature of the number π, we shall have some sort of an answer. We may suggest that there are people who find a fascination in this *furor arithmeticus*, and any mathematician might take an interest in obtaining other forms of the series, which, by reason of the simplicity of the denominators of the fractions or by the rapidity with which they converge, improve on forms already obtained and used.

Much ingenuity has been expended on this sort of search. Some samples are given:

DASE used $\dfrac{\pi}{4} = \tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{8}.$

MACHIN found and used $\dfrac{\pi}{4} = 4\tan^{-1}\dfrac{1}{5} - \tan^{-1}\dfrac{1}{239}.$

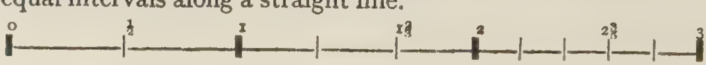
This,

$$\frac{\pi}{4} = 22\tan^{-1}\frac{1}{28} + 2\tan^{-1}\frac{1}{443} - 5\tan^{-1}\frac{1}{1393} - 10\tan^{-1}\frac{1}{11018}$$

(due to ESCOTT), is a marvel of ingenuity. It would be very rapidly convergent but, involving divisions unsuited to mental work, would be cumbrous in working.

The Nature of the Number π.—In order that what follows may be quite clear, we must digress a little and consider the classification of numbers.

We can construct a number scale, as we can graduate a ruler. To represent integers or whole numbers we can mark off points at equal intervals along a straight line.



If these intervals are divided into 2, 3, 5, etc., equal parts we get points corresponding to fractions. All vulgar and decimal fractions can be thus located on the number scale.

Integers and fractions are the two subdivisions of the class "commensurable numbers." The term "commensurable" is used because any two can be regarded as multiples of a common measure thus $\frac{8}{5}$ and $\frac{2}{3}$ are both whole numbers of multiples of the common measure—"one-fifteenth of a unit."

But there are other numbers ("incommensurables,") which have a place in the number scale, but whose positions cannot be located by equal subdivision. Such numbers as $\sqrt{2}$, $\log_{10} 3$, $\tan 10°$ are incommensurables, i.e. they have no common measure with unity. They can be obtained as accurately as required for any practical purpose, but they are neither integers nor fractions, and their exact arithmetical value cannot be obtained.

Incommensurables are divided into two classes—irrationals (or surds) and transcendentals. Irrationals such as $\sqrt{2}$, $\sqrt[3]{2}$, $1 + \sqrt{2}$ are roots of algebraic equations of the type

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots + kx + l = 0 \quad \ldots\ldots (1)$$

$\sqrt{2}$ is a root of $x^2 = 2$, $\sqrt[3]{2}$ of $x^3 = 2$, $1 + \sqrt{2}$ of $x^2 - 2x = 1$.

Transcendentals, such as $\log_{10} 3$ or $\tan 10°$, are not roots of equations of this type.

There are other numbers (complex numbers). They are of the form $m + n\sqrt{-1}$, and have no place in the number scale; they may be roots of equations of the type (1), e.g. $1 + 2\sqrt{-1}$ is a root of $x^2 - 2x + 3 = 0$. To these numbers no arithmetical approximation can be obtained. They are discussed more fully in Chapter XIV.

Now the nature of the number π is important in connexion with the problem of squaring the circle in this way:—that if it could be shown to be a commensurable number or a surd derived from commensurable numbers by taking square roots, then the problem as stated could be solved. If π were $3\frac{16}{113}$, $\sqrt{10}$, $3 + \sqrt{\frac{1}{50}}$, $\sqrt{10 - \frac{1}{10\sqrt{6}}}$, etc., then it could be obtained with circle and straight-line construction.

The first contribution to this line of treatment was made by LAMBERT (Johann Heinrich; *b.* Mülhausen, 1728; *d.* Berlin, 1777). In 1768 he published a memoir proving that π is incommensurable.

In 1803 LEGENDRE (*b.* Toulouse, 1752; *d.* Paris, 1833) extended the proof to the incommensurability of π^2.

EULER showed that π satisfied the equation $1 + e^{\pi\sqrt{-1}} = 0$, a remarkable equation connecting, in a simple form, 1, the basis of all counting; e, the base of natural logarithms (*see* p. 181); $\sqrt{-1}$ the simplest complex number, and π.

In 1873 HERMITE (*b.* Lorraine, 1822; *d.* Paris, 1901) proved that e is transcendental, and LINDEMANN, in 1882, using this and $1 + e^{\pi\sqrt{-1}} = 0$, proved π to be transcendental.

Until π was proved to be transcendental, investigation inevitably continued; and as it was not impossible that by working out its value something of its nature might be discovered, we must ascribe to something more than *furor arithmeticus* the labours of Sharp, Rutherford, Shanks, and other computers.

An Approximate Construction.—KOCHAUSKY (1685) gave the following construction for π:

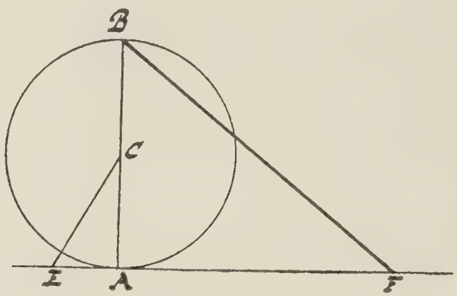


FIG. 30

Let C be the centre and BA a diameter of a circle of unit radius.
Draw EAF, the tangent at A.
Make $\angle$ ACE = 30° and make EF = 3 units.
Then BF is nearly π units.

$$\text{EA} = \frac{1}{\sqrt{3}};$$

$$\therefore \quad \text{AF} = 3 - \frac{\sqrt{3}}{3};$$

$$\therefore \quad \text{BF}^2 = \text{AB}^2 + \text{AF}^2 = 4 + 9\tfrac{1}{3} - 2\sqrt{3},$$

i.e.

$$\begin{aligned} \text{BF} &= \sqrt{13\tfrac{1}{3} - 2\sqrt{3}} \\ &= \sqrt{\{13{\cdot}33333 - 3{\cdot}46410\}} \\ &= \sqrt{9{\cdot}86923} \\ &= 3{\cdot}1415, \text{ nearly.} \end{aligned}$$

But this differs from the approximate constructions given on pp. 44–47, as being based on an arithmetical rather than a geometrical consideration. It does not start with a semi-circumference and reduce it to a straight line of approximately the same length; it builds up in geometrical form a number to approximate to the numerical value of π.

Thus it is ingenious in the same way (but without being as useful) as the approximations.

$$\pi = 3 + \tfrac{1}{8} + \tfrac{1}{60} \quad = 3{\cdot}141\dot{6},$$
$$\pi = 3\tfrac{1}{7}\,(1 - 0{\cdot}0004) = 3{\cdot}14160 \text{ (nearly)},$$
$$\frac{1}{\pi} = \frac{3}{10} + \frac{1}{60} + \frac{1}{600} = \ {\cdot}318\dot{3},$$
$$\frac{1}{\pi} = \frac{1}{3} + \frac{1}{200} - \frac{1}{50} = \ {\cdot}318\dot{3}\ \left(\frac{1}{\pi} \text{ being } {\cdot}31830989\ \ldots\right)$$

Occurrence of π.—As Rouse Ball points out, it is not merely in the mensuration of the circle that π turns up; it occurs in mathematical analysis in various ways; e.g.:

If a stick of length l is dropped on to a plane ruled with parallel straight lines equally spaced at distances a, the probability that it will touch a line is $2l / \pi a$. $(l < a)$

If two numbers are written down at random, the probability that they are prime to each other is $6 / \pi^2$.

The following formulæ and series involve π:

The surface of a sphere $= 4\pi r^2$,
The volume of a sphere $= \tfrac{4}{3}\pi r^3$,
The area of an ellipse $= \pi ab$,
The volume of a spheroid
(such as a Rugby football) $= \tfrac{4}{3}\pi ab^2.$,

$$\frac{1}{6}\pi^2 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$
$$\frac{1}{8}\pi^2 = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$
$$\frac{1}{12}\pi^2 = \frac{1}{1.2} + \frac{1}{2.4} + \frac{1}{3.6} + \frac{1}{4.8} + \cdots$$
$$\frac{1}{90}\pi^4 = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$
$$\frac{1}{96}\pi^4 = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$$
$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\cdots$$
$$\cos\theta = \left(1 - \frac{2^2\theta^2}{\pi^2}\right)\left(1 - \frac{2^2\theta^2}{3^2\pi^2}\right)\left(1 - \frac{2^2\theta^2}{5^2\pi^2}\right)\cdots$$
$$\frac{16}{3\pi} = \frac{4\cdot5}{2\cdot4} \times \frac{5\cdot6}{4\cdot6} \times \frac{6\cdot7}{6\cdot8} \times \frac{7\cdot8}{8\cdot10} \times \cdots$$

The coefficient of x^n in $(1 + x)\left(1 + \dfrac{x}{2^2}\right)\left(1 + \dfrac{x}{3^2}\right)\cdots$

is $\dfrac{\pi^{2n}}{(2n+1)!}$.

CHAPTER V

AN INTRODUCTION TO SOME CURVES

Their Construction and Properties.—The practice in schools of studying somewhat exhaustively the properties of the circle and of neglecting almost entirely all other curves is no doubt a legacy of our servitude to Euclid. We should, however, remember that Euclid's geometry professed to be no more than the "Elements" of the subject. Although the circle is unique in its symmetry and of fundamental importance, it is only one of many curves.

The ellipse figures in planetary orbits and in some arches; the parabola is taken as the ideal path of a projectile, such as a cricket ball in flight; the catenary gives the curved lines in the suspension bridge; the hyperbola is used in sound-ranging, and the sine curve is involved in the theory of pendulums, of times of sunrise, of tides and of wave motion generally.

These and other curves the engineer employs and the citizen with any geometrical vision must notice about him. Even before Euclid's time, the Greeks were discovering and investigating some of them. And the student whose studies do not touch them is likely to have his geometrical ideas seriously narrowed. Even if his detailed study is restricted to the circle, he will study that to better purpose if he gets a glimpse of the properties and of the mathematical investigation of other curves.

This chapter will deal with some curves, with some simple properties of them, with methods of construction, and with some consideration of their equations.

The parabola and hyperbola are commonly plotted in graphical work, but the pupil's knowledge of them need not be limited to recognizing that certain simple functions can be represented by these curves. He can learn some of their properties. Many of them can be plotted quite simply; and the plotting provides good practice for inculcating the idea of a "locus." For the construction of some curves simple mechanical apparatus can be devised. The mechanical devices are made up of details to produce constraints corresponding to certain data:

(1) If a point P is to move at a constant distance from a fixed point A, it can be traced by a tracing point at one end P of a rod PA, of which the other end is pivoted at A.

(2) If a point P is to move along a straight line AB, the AB

of the apparatus must have a slot or groove, along which a peg P is guided.

(3) If a straight line PQ is to pass through a fixed point A, a rod PQ will have a slot which guides it along a fixed peg A.

(4) If a straight line AB is to move so as always to be parallel to a given straight line CD, four rods pivoted to form a parallelogram ABCD will provide for it.

The Circle.—In a board make two straight grooves XOX_1, YOY_1 at right angles.

Take a rod AB fitted with two pegs A and B and a tracing point P midway between; OX, OX_1, OY and OY_1 must each be $>$ AB. A will move along XX_1 and B along YY_1.

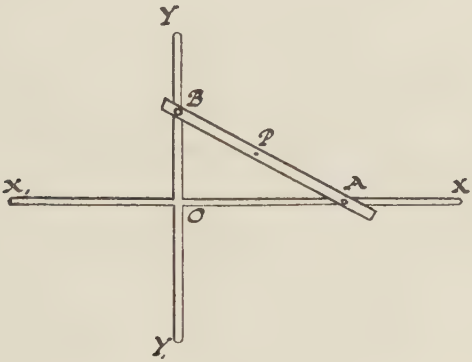


FIG. 31

Start with A at O and B towards Y, let A move towards X and B towards O. As B passes O, A reaches its farthest point. As B moves on towards Y_1, A returns towards O. When A reaches O, B is at its farthest point. A now moves on towards X_1 and B returns. When B has again reached its first position, a complete curve has been traced by P, a circle.

Mechanisms of this kind are called **trammels.** The trammel being a rod moving in a plane under two constraints loses two of its degrees of freedom and a point on it describes a locus (*see* Ch. XX).

Geometrical Counterpart by Plotting.—Let YOY_1 and XOX_1 be two straight lines at right angles (Fig. 32).

Let A be any point on XX_1. With A as centre and AB as radius draw an arc cutting YY_1 in B_1 and B_2. Bisect B_1A and B_2A at P_1 and P_2. P_1 and P_2 are points on the locus.

Or. Take a piece of paper or a straight-edge, marked with three points A, P, B such that AP = PB. Place it so that B is on YY_1 and A on XX_1. Prick the positions of P.

The proof is a familiar one.

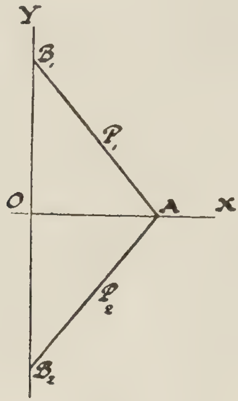


FIG. 32

The Ellipse.—If P is taken so that AP ≠ PB, P traces an Ellipse.

Let PA < PB. Then PB is the semi-major axis and PA the semi-minor axis.

Proof.—Draw PN perpendicular to OX.

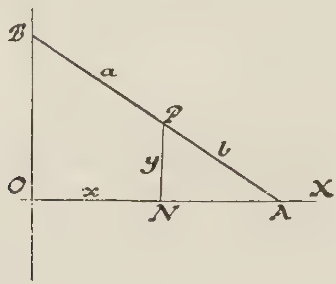


FIG. 33

Let BP $= a$; PA $= b$; ON $= x$; PN $= y$.

By similar triangles,

$$\mathrm{AN} = \frac{bx}{a};$$

and in $\triangle$ APN,

$$b^2 = y^2 + \mathrm{NA}^2$$
$$= y^2 + \frac{b^2x^2}{a^2},$$

i.e.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots\ldots\ldots\ldots \text{(I)}$$

And this is the equation of an ellipse.

Corresponding Definition of an Ellipse.—Let us see to what geometrical definition of an ellipse this equation corresponds.

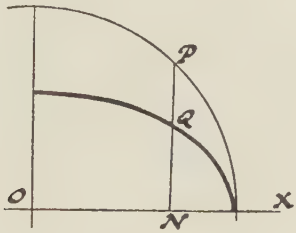


FIG. 34

Let P be a point (x, y) on a circle of centre O and radius a.

Then in the $\triangle$ PON,

$$x^2 + y^2 = a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II)}$$

Let Q be a point in PN so that $\frac{\text{QN}}{\text{PN}}$ is a constant $= \frac{b}{a}$.

Then if x, k are the co-ordinates of Q,

$$k = \frac{by}{a}, \text{ i.e. } y = \frac{ak}{b};$$

by substitution in (II)

$$x^2 + \frac{a^2k^2}{b^2} = a^2;$$

i.e.

$$\frac{x^2}{a^2} + \frac{k^2}{b^2} = 1;$$

and since k is the ordinate of Q, the equation would ordinarily be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Thus we get the equation (I) from a geometrical method which is equivalent to regarding an ellipse as the orthogonal projection of a circle.

The points Q can be plotted as follows:

Draw a circle, centre O, having as radius the semi-major axis of the required ellipse. Take ordinates, as PN, and divide them at Q so that QN : PN :: minor axis : major axis. Then the points Q are on the ellipse.

A more convenient way is to draw two concentric circles of radii a and b (semi-major and semi-minor axes).

Draw any radius OSP.

Through S and P draw lines parallel to OX and OY, meeting in Q. Q is a point on the ellipse. For by similar triangles,

$$QN : PN :: SO : OP :: b : a.$$

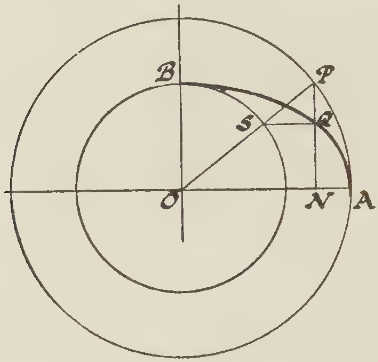


FIG. 35

Discussion of the Cartesian Equation of the Ellipse.—We thus see that accepting the projection definition of the ellipse we could deduce the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as typical. Some discussion of the equation will show something of the methods of Cartesian geometry and encourage us to use them when occasions arise.

If $-x$ is substituted for x, or $-y$ for y the equation is unchanged, i.e. the curve is symmetrical about both axes (*see* p. 131).

When $x = 0$, $y = \pm b$. These co-ordinates give the extremities of the minor axis.

When $y = 0$, $x = \pm a$ and these give the extremities of the major axis.

Again $x = \pm a\sqrt{1 - \frac{y^2}{b^2}}$; $\therefore$ if $y > b$ or $< -b$, x is imaginary, i.e. the curve must lie between $y = b$ and $y = -b$; similarly it lies between $x = a$ and $x = -a$. It lies therefore entirely within a rectangle whose sides are the lengths of the axes.

The equation is of the 2nd degree, i.e. a straight line will meet it in two points—real, coincident or imaginary. Another ellipse or circle may intersect it in four points (*ee* p. 172).

The Area of an Ellipse.—The same definition enables us to find the area of an ellipse.

Let an ordinate perpendicular to the major axis A_1A meet the circle, ellipse, and axis in P, Q and N and a near ordinate meet them in R, S, and M.

Then if the ordinates are sufficiently close the error in regarding RMNP and SMNQ as rectangles is negligible.

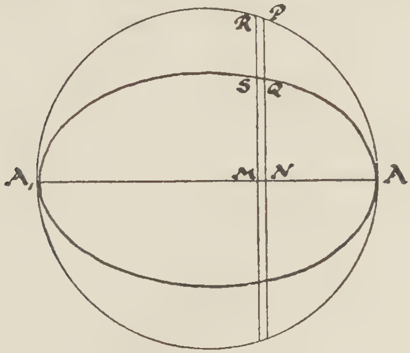


FIG. 36

And the area $\dfrac{\text{SQMN}}{\text{RMNP}} = \dfrac{\text{QN}}{\text{PN}} = \dfrac{\text{SM}}{\text{RM}} = \dfrac{a}{b}$.

By dividing the figure up into such strips it is seen that the area of the ellipse : the area of the circle :: $b : a$;

$\therefore$ area of ellipse $= \pi a^2 \times b / a = \pi ab$.

In the same way the volume of a spheroid (like the Rugby football) can be shown to be $\frac{4}{3}\pi ab^2$.

The Conic Sections.—Another definition and another construction can be given for the ellipse, due to the investigations of APOLLONIUS (260–200 B.C.)

In Fig. 37 let ROEAG and QODAF be the lines in which a double right circular cone, with common vertex O, is cut by a plane through its axis.

Let APA_1 be a plane section of the cone ; DSE and FGH sections of spheres which touch the plane APA_1 and the cone ; DE and FG the diameters of the circles of contact.

Let S and H be the points where the spheres touch the plane APA_1 ; and let X and X_1 be the points where AA_1 meets DE and GF.

Let XK be perpendicular to XX_1 in the plane APA_1, and from P let PK be perpendicular to XK.

Then, using the equality of tangents to a sphere from an external point, we can prove that SP : PK :: SA : AX ; and SA : AX does not depend on the position of P ; it is constant for every point on the periphery of the plane section APA_1.

The ratio SA : AX is called the **eccentricity**, as we shall see later, and is denoted by the symbol e.

Now SA = AE and $\therefore$ SA : AX :: EA : AX.

If $e = 1$, EA = AX, $\therefore$ $\angle$ AXE = XEA = OED = ODE ; and AX is parallel to QO ; the points H, A_1, X_1 are at infinity. The section is a parabola (*see* p. 76).

If $e < 1$, EA < AX, $\angle$ EXA < XEA < ODE, and the section cuts QO below O ; the section is an ellipse.

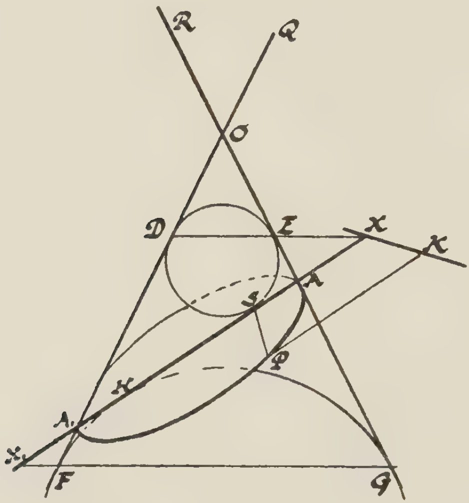


FIG. 37

If $e > 1$, EA > AX, and $\angle$ EXA > ODE $\therefore$ the section meets OQ above O ; and H, A_1, X_1 are connected with the upper cone. The section is a hyperbola.

Special case : if $e = 0$, EA = 0, A coincides with E, and the section is perpendicular to the axis of the double cone. It is a circle.

There are some other special cases.

These sections are called Conic Sections. They can be defined as loci of points such that the distance from a fixed point S (the focus) bears a constant ratio to the distance from a fixed line XK (the directrix).

Plotting Conic Sections.—To plot the Conic Sections :

Let S be the fixed point and LK the fixed line. Draw SX perpendicular to LK.

Along SX there will be two points (one in the parabola) A, A_1 (called vertices), such that SA : AX = e (or $-e$).

Draw lines l_1, l_2, l_3 . . . parallel to LK at distances greater than XA. With centre S, draw circles c_1, c_2, c_3, . . . so that the

radius of c_n : distance of l_n from LK :: e : 1. Then the intersection of l_n and c_n gives two points on the locus.

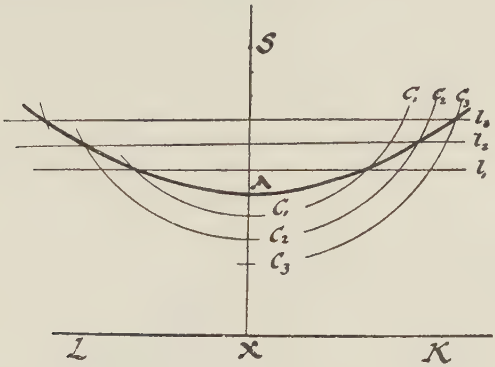


FIG. 38

All these sections have equations of the 2nd degree.

Take S as the origin, SX as the axis of x. Then let any point on the curve be (x, y). (Fig. 39.)

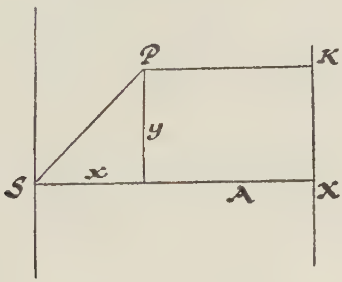


FIG. 39

$$SP : PK :: e : 1$$

or $$SP = ePK$$

Now, $$SP = \sqrt{x^2 + y^2},$$

and $$PK = c - x, \text{ where } SX = c;$$

$$\therefore \quad x^2 + y^2 = e^2(c - x)^2,$$

and this is of the 2nd degree.

Conversely it could be shown that every equation of the 2nd degree represents a conic section.

Properties of the Ellipse.—We will now deduce some properties of the ellipse.

Let S and H be foci; A, A_1 vertices; XK and X_1K_1 the directrices; and C the mid-point of SH be the centre.

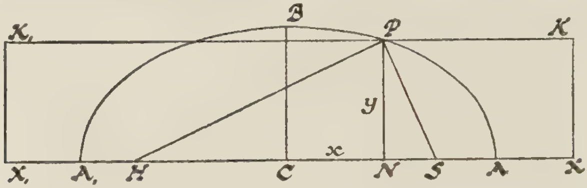


FIG. 40

Let P be any point on the circumference, and let K_1PK be perpendicular to the directrices.

Then $\qquad SP = ePK$

and $\qquad HP = ePK_1;$

$$\therefore \quad SP + HP = e(PK + PK_1)$$
$$= e(XX_1);$$
$$\therefore \quad SP + HP \text{ is a constant.}$$

This gives us another definition:—that an ellipse is the locus of a point P, such that the sum of its distances from two fixed points S and H is constant; the analogous definition for a hyperbola is obtained by substituting "difference" for "sum."

Mechanical Construction of the Conic Sections.—These definitions suggest mechanical methods of construction:

(1) For the ellipse, fasten two pegs S and H in a board; round them put a loop of string. Place a tracing point P in the loop and move it so as to keep the string always taut. Then P traces out an ellipse.

(2) For the hyperbola, take a rod HJ pivoted at H; to J attach a piece of string and fix the other end at S. Place a tracing point P against the string on the side remote from HJ, so as to keep the string taut. P traces out a hyperbola.

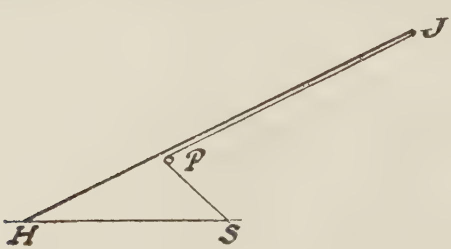


FIG. 41

For $\quad HP - PS = (HP + PJ) - (SP + PJ)$
$\qquad = $ (the length of the rod) $-$ (the length of the string).

Now, in the case of the ellipse, when P coincides with A (Fig. 40), HP + SP becomes

$$HA + AS = HA + HA_1 = AA_1 = 2CA\ ;$$

$$\therefore\quad \text{the constant} = 2CA = 2a, \text{ where } CA = a,$$

and of course a is the semi-major axis.

And, as we have seen,

$$HP + PS = e(XX_1)$$

$$\therefore\quad XX_1 = \frac{2a}{e}\ ;$$

$$\therefore\quad CX = \frac{a}{e}\ ; \quad \ldots\ldots\ldots\ldots \quad \text{(I)}$$

$$\therefore\quad AX = CX - CA = a\left(\frac{1}{e} - 1\right),$$

and

$$SA = eAX = a(1 - e)\ ;$$

$$\therefore\quad CS = CA - AS = ae, \quad \ldots\ldots\ldots \quad \text{(II)}$$

i.e. the ratio $CS : CA = e : 1$,

and this relation explains the word "eccentricity" (*ek* = from, *kentron* = centre), for CS measures e for a given distance CA, and CS is the distance of S **from the Centre.**

Again, when P is at B on the $\perp$ to XX_1

$$BH = BS \text{ by symmetry,}$$

and

$$\therefore\quad BS = a\ ;$$

$\therefore$ if b is the semi-axis minor,

$$b^2 = BC^2 = BS^2 - CS^2 = a^2 - a^2e^2 = a^2(1 - e^2) \quad . \ . \quad \text{(III)}$$

We have now equations connecting the eccentricity and the lengths of the major and minor axes. Incidentally we see how to find the foci for an ellipse of given semi-axes, CA and CB: with B as centre and radius equal to CA we draw a circle cutting AA_1 in S and H, and so we can now use the looped-string method to draw an ellipse having its axes of a required length.

We have also several definitions of the ellipse:

(1) A projection of the circle—corresponding to the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(2) A section of a cone.

(3) The locus of P, such that its distance from a fixed point S is e times its distance from a fixed line XK.

(4) The locus of a point P, such that the sum of its distances from two fixed points S and H is constant.

Definitions (2) and (3) have been shown to be equivalent. It remains to show that (3) and (4) can be made to agree with definition (1), i.e. that they are equivalent to the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Take definition (3):

Let CA and CB be the axes of x and y (Fig. 40 or 43). Draw PN perpedicular to AA_1 and let P be (x, y).

Then $$SP^2 = e^2\,PK^2,$$
i.e. $$PN^2 + NS^2 = e^2(CX - CN)^2,$$
$$y^2 + (ae \sim x)^2 = e^2\left(\frac{a}{e} - x\right)^2 = a^2 + e^2x^2 - 2\,aex,$$
i.e. $$x^2(1 - e^2) + y^2 = a^2(1 - e^2),$$
$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$
But $$a^2(1 - e^2) = b^2, \text{ by equation (III)},$$
i.e. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Again, the projection property gives us that for P, $x = a \cos \theta$, $y = b \sin \theta$. This can be seen to satisfy the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

These co-ordinates can be obtained directly thus:

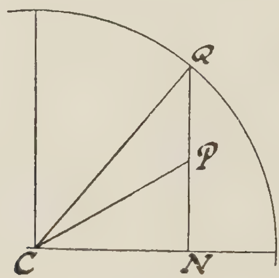


FIG. 42

Let Q be a point on the circumference of the circle, QN an ordinate, and P a point in QN such that PN : QN :: $b : a$.

Then if $\angle$ QCN is θ,

$$CN = a \cos \theta \text{ and } PN = \frac{b}{a}\,QN = \frac{b}{a} \cdot a \sin \theta = b \sin \theta.$$

We have seen that definitions (1) and (3) agree; (4) can be deduced from (3). Using Fig. 43 we get:—

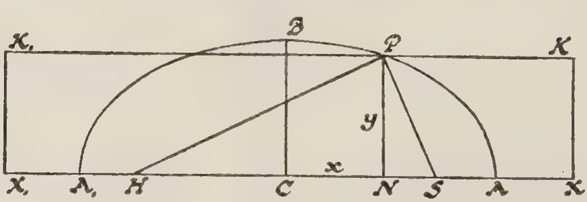


FIG. 43

$$\begin{aligned}HP^2 &= (HC + CN)^2 + PN^2\\ &= (ae + x)^2 + y^2 = (ae + a\cos\theta)^2 + b^2\sin^2\theta\\ &= a^2e^2 + 2a^2e\cos\theta + a^2\cos^2\theta + a^2(1 - e^2)\sin^2\theta\\ &= a^2e^2\cos^2\theta + 2a^2e\cos\theta + a^2\\ &= a^2(e\cos\theta + 1)^2;\\ \therefore\quad HP &= a(1 + e\cos\theta).\end{aligned}$$

And similarly

$$SP = a(1 - e\cos\theta),$$

i.e. $$SP + PH = 2a.$$

The reader will find it an interesting exercise to go through the hyperbola in the same way.

From the property $SP + PH = 2a$, which is the property corresponding to the simple and well-known mechanical construction given on p. 72, some other interesting properties can be deduced.

Take AA_1 as fixed; as CS decreases, i.e. as the eccentricity decreases, S and H approach; and finally when $e = 0$, S and H coincide at C, and SP + PH becomes 2CP and $\therefore$ CP $= a$; i.e. if we define the circle as a Conic Section of zero eccentricity, we can deduce the property that all radii are equal.

Again, if R is outside an ellipse (Fig. 44),

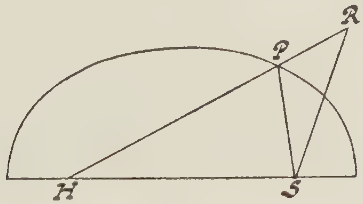


FIG. 44

$$HR + RS > 2a.$$

For let HR meet the ellipse in P,

Then $HR + RS = HP + (PR + RS) > HP + PS > 2a$.

Now, consider a tangent TPt at P (Fig. 45). Every point

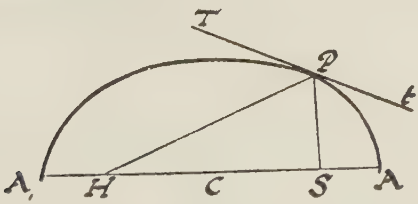


FIG. 45

except P is outside the ellipse $\therefore$ HP + PS is the minimum route from H to TPt and on to S.

Therefore, by a well-known rider, $\angle$ TPH = $\angle$ tPS. This is a tangent property for the ellipse; when H and S coincide it becomes a property of a tangent to a circle. But when H and S coincide, HP and SP coincide as a radius, and $\angle$ s HPT and SPt are adjacent, and, being equal, they are right angles; $\therefore$ the tangent at a point of a circle is perpendicular to the radius through the point.

The Parabola a Particular Case of the Ellipse.—Again, when $e = 1$, CS = CA; but this can only be so if C has moved away

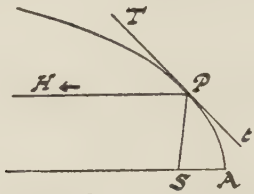


FIG. 46

to infinity, and in that case H has also moved to infinity and PH is parallel to the axis AS.

But when $e = 1$ the ellipse becomes a parabola; and this property of the tangent of the parabola is used in searchlights.

For if S is a point-source of light and the point P is on a parabolic reflector of which S is the focus, a beam of light SP reflected at P will go along PH where $\angle$ TPH = $\angle$ tPS; i.e. all incident beams from S are reflected parallel to the axis. But in a searchlight the source of light is not a geometrical point, and consequently rays from points near to S are not reflected parallel to AS. Even so, the use of the parabolic reflector prevents dispersion and ensures intensity of beam.

The Path of a Projectile.—The parabola is the path of a projectile *in vacuo*.

Consider a shell fired with muzzle-velocity of 800 feet per

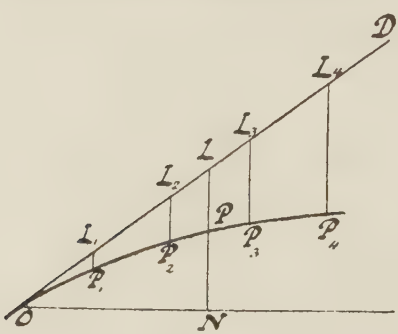


FIG. 47

second at an angle of 45°. If it were not for the attraction of the earth its path would be a straight line at 45° to the horizontal.

Let OD represent this path, and let L_1, L_2, L_3 . . . represent the positions after 1, 2, 3 . . . seconds. But the fall due to gravity is about $16t^2$ feet in t seconds. If L_1P_1, L_2P_2, L_3P_3 . . . represent vertical distances of 16, 64, 144 . . . feet, then $OP_1P_2P_3$ would be the path, taking gravity into account but neglecting air resistance.

Let P be the position on the path after time t, and LPN an ordinate meeting OD in L and the horizontal through O in N. Take O as the origin and let P be (x, y). Then, in feet,

$$\mathrm{OL} = 800t,$$
$$\mathrm{ON} = \mathrm{LN} = 400\sqrt{2}t,$$
$$\mathrm{LP} = 16t^2;$$
$$\therefore\quad x = 400\sqrt{2}t, \quad\ldots\ldots\ldots\ldots\quad (1)$$
$$y = 400\sqrt{2}t - 16t^2 \quad\ldots\ldots\ldots\ldots\quad (2)$$

From (1) $$t = \frac{x}{400\sqrt{2}}.$$

Substituting in (2):

$$y = x - \frac{x^2}{20000} \quad\ldots\ldots\ldots\ldots\quad (\mathrm{I})$$

This is the equation of a parabola as our experience of graphs shows us.

[The reader must be warned that it is more correct to say that the path of a projectile *in vacuo* is part of an ellipse of which the earth's centre is a distant focus. We have taken L_1P_1, L_2P_2, etc., as parallel, whereas they are really parts of radii of the earth.]

(I) may be written:

$$y = -\frac{1}{20000}(x^2 - 20000x + 10^8) + \frac{10^8}{20000}$$
$$= 5000 - \frac{1}{20000}(x - 10000)^2.$$

When $x = 10000$, y has its maximum value. That is, the highest altitude the projectile reaches is 5,000 feet and half the range is 10,000 feet, i.e. the range *in vacuo* for a gun elevation of 45° is 20,000 feet.

It should be noted that Fig. 47 suggests another way of plotting a parabola.

We have not room here for further discussion of the Conic Sections, except to mention the use of the hyperbola in sound-ranging.

The Hyperbola in Sound-Ranging.—As we saw on p. 72, the hyperbola may be defined as the locus of P, such that $\mathrm{PH} \sim \mathrm{PS}$ is constant, where H and S are fixed points—the foci.

Now let H and S be two observation posts and P a gun. When

P fires, recording instruments at H and S register the time when the report is heard at H and S respectively.

For the recorded difference in the times, HP $\sim$ HS is calculable and a hyperbola can be plotted on which P lies. Other observation posts working with H and S will obtain time-differences enabling

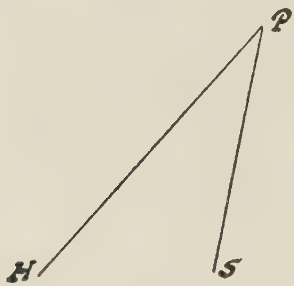


FIG. 48

other hyperbola-loci of P to be plotted. Where the loci intersect is the position of P.

Now, if HP = PS the hyperbola becomes a straight line—the right bisector of HS; and if HP $\sim$ PS is small relatively to the distances involved, the hyperbola approximates to a straight line in the neighbourhood of P. In practice, observers take advantage of this by arranging a number of observation posts—usually about six—at approximately equal distances from the conjectured position of P, and so simplify their work and check results.

Mechanical Construction of the Parabola.—Arrange a ⊤-square to slide along XL, the straight edge of a board. To R attach a string of the same length as RK, the leg of the ⊤, and fix the end

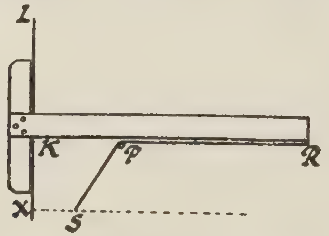


FIG. 49

S at a point on the board. Move a tracing point P so that it keeps the string taut, and so that the part PR is in contact with KR.

Then $$KP + PR = RP + PS;$$
$$\therefore \quad SP = PK,$$

and P traces out a parabola of which S is the focus and XL the directrix.

Discussion of the Equation of the Cissoid.—In Chapter III we have seen how the Greeks employed the hyperbola, parabola, and other curves for the solution of certain problems. We have seen (pp. 39 and 40) how the conchoid and cissoid can be plotted. The equation of the cissoid with O as origin was found (p. 41) to be

$$x^3 = (2a - x)\,y^2,$$

i.e.
$$y^2 = \frac{x^3}{2a - x}.$$

If x is negative, say $-b$,

$$y^2 = \frac{-b^3}{2a + b},$$

and y is imaginary, therefore the curve lies wholly on the positive side of OY.

If $0 < x < 2a$; $\dfrac{x^3}{2a - x}$ is positive and $y = \pm\sqrt{\dfrac{x^3}{2a - x}}$, i.e.

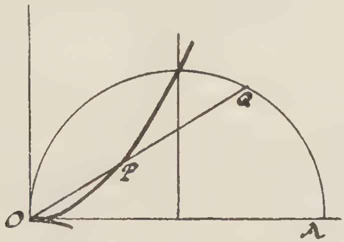


FIG. 50

two real values of y are obtained equal in magnitude and opposite in sign, therefore two positions of P are found symmetrical about OA.

If $x > 2a$, $2a - x$ is negative and y is again imaginary, therefore the curve lies entirely between the axis of y and the ordinate at A.

Again, when $x = 0$, $y = 0$ and therefore O is a point on the curve. As x increases from 0 to $2a$, the numerator x^3 increases and the denominator $2a - x$ decreases; therefore the numerical value of y increases, and increases more rapidly than x^3; and the part of the curve in the first quadrant, therefore, lies between $y = x^3$ and the axis of y. Similarly the part in the fourth quadrant lies between $y = -x^3$ and the axis of y.

When $x = 2a, y = \pm\sqrt{\dfrac{8a^3}{0}}$ and is $\pm\infty$, therefore the ordinate at A is an asymptote.

Finally, since the equation is of the 3rd degree, a straight line may cut it in three, but not more than three, points.

These considerations would give us a good idea of the shape

of the curve represented by $x^3 = (2a - x)y^2$ without plotting, and they agree, as they should do, with the shape obtained by plotting.

Mechanical Construction of the Cissoid.—Let OPSQ and SG form a rigid ⊤ with OSG a right angle.

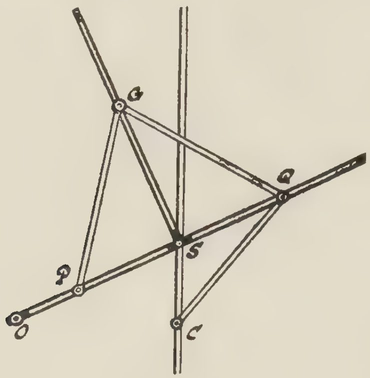


FIG. 51

Let both arms be slotted.

Let O be a pivot fixed into a board, C a fixed point in the board, and CS a groove perpendicular to OC. Let CQ be a rod equal in length to OC, and PG and GQ equal rods pivoted at G, so that G moves along SG and P along OPQ; and let GQ and CQ be pivoted at Q, so that Q slides along OPQ. Then Q traces out a circle of centre C, and passing through O; SC is part of a diameter of the circle; PS = SQ, and P traces out the cissoid.

The Conchoid of Nicomedes.—The conchoid can be traced by the use of a trammel thus:

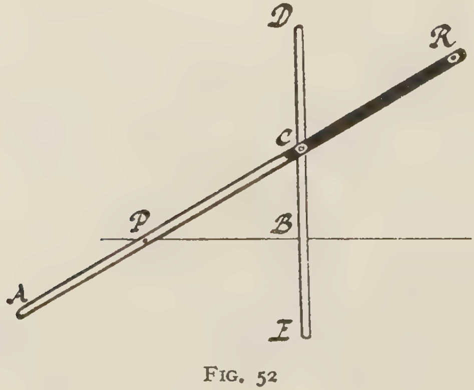


FIG. 52

Take a board with a peg at P and a groove DBE; take a rod ACR with a peg C, a slot from A towards C, and a tracing point R. Place the rod so that P lies in the slot of the rod and C in the groove of the board; R traces out the conchoid.

Plotting the Conchoid.—Most of the curves that can be traced by a trammel can be plotted very simply by the use of a straight edge or even a piece of paper.

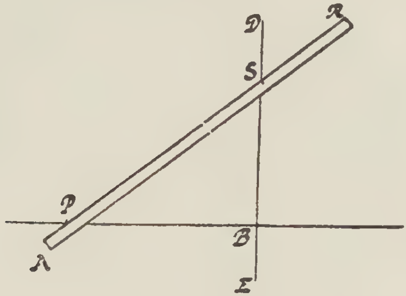


FIG. 53

On paper mark a point P and a straight line DE. Take a ruler or piece of paper with a straight edge ASR; mark fixed points S and R. Place ASR so that S is on DE and P on AS; prick on the paper the point where R lies. The points pricked can then be joined up by a pencil line.

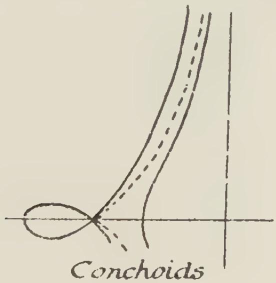


FIG. 54

R can be taken between S and A; and according as SR $<, =, >$ PB, the perpendicular from P to DBE, the form of the curve varies in the neighbourhood of P, a variation of form that will be observed in other curves.

The equation $(x^2 + y^2)(x - a)^2 = b^2x^2$ can be obtained and discussed in the same way as that of the cissoid. In the process it will be found that, to satisfy certain analytical considerations, the conchoid must be regarded as consisting of two conjugate branches on

opposite sides of DE. The separate branches are shown in Figs. 14 and 54.

Other conchoids can be plotted as follows:

(1) Take a circle of centre O and fixed radius OA. At A draw the tangent AD. Through O draw a line cutting the circle in C and the tangent in D. Through C and D draw lines parallel to AD and OA. P, the intersection of these lines, is a point on the conchoid.

(2) Take a circle of centre O and fixed radius OA. Through O draw a line OD perpendicular to OA. Through A draw ACD cutting the circle in C and OD in D. Through C and D draw lines CP and DP parallel to OD and OA. The locus of P is a conchoid.

If the construction for plotting the conchoid given on p. 81 is varied by substituting the circumference of a circle for the straight line DBE as the locus of S (Fig. 53), other curves will be obtained. In the same way the circumference of a circle can be substituted for one or both of the straight lines OA and OB (Fig. 32, p. 66).

The Limaçons and Cardioid.—If the straight line BD of Fig. 53 is replaced by a circle passing through P, some other curves are

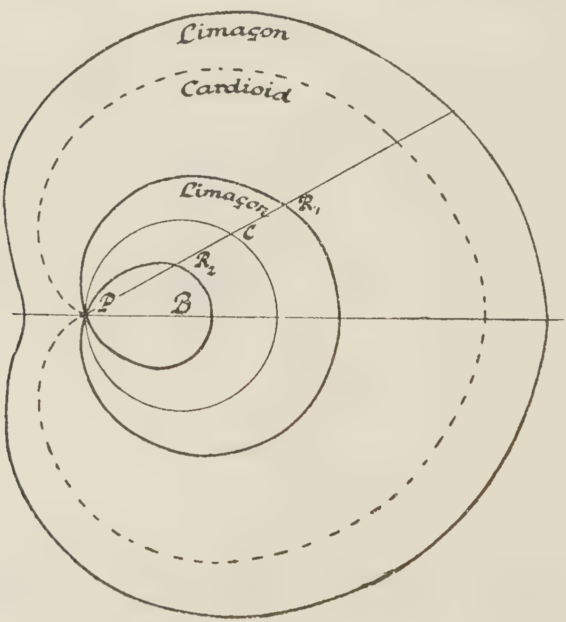


Fig. 55

obtained:—**Pascal's Limaçons** (Fr., *limaçon*: *a snail*) and the **Cardioid** (*the heart-shaped curve*). They can be plotted as follows:

Let P be a fixed and C a variable point on a circle of

diameter d. Join PC and cut off $CR_1 = CR_2$ of constant length. Then the points R_1 and R_2 are on the locus.

If $CR_1 > d$ we get the limaçon without a loop; if $CR_1 = d$ we get the cardioid; if $CR_1 < d$ we get the limaçon with a loop.

These curves can also be plotted by using a ruler in the same sort of way as for a conchoid.

The mechanical construction can be carried out by the modification of the trammel of Fig. 11 which is shewn in Fig. 56.

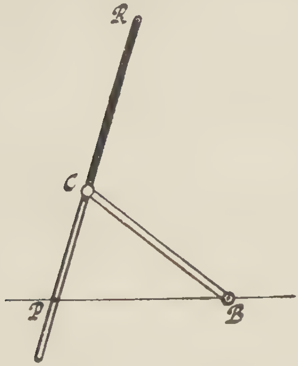


FIG. 56

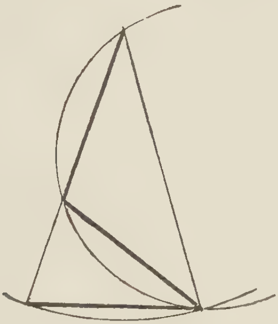


FIG. 57

P is a peg, PCR a slotted rod with a tracing point at R. B is a fixed point. BC is a rod equal in length to BP and pivoted with CR at C. C traces out a circle, and R the other curves.

It should be noted that if CR = CB = BP, the linkage is the trisectrix linkage of p. 36; but it is arranged for a different motion; and it may be noted that in one special position, viz. when BR = RP, we obtain the figure for the construction of the isosceles triangle having each angle at the base double of the vertical angle. (Fig. 57.)

Equations of Limaçon and Cardioid.—The simplest form in which to write them is in polar co-ordinates.

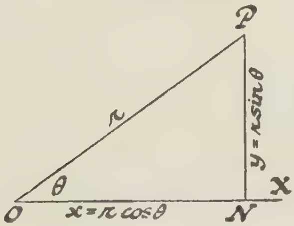


FIG. 58

If OX is a fixed line and OP makes with it a variable angle

θ, and OP is a length r, then the polar co-ordinates of P are r and θ. These two co-ordinates fix the position of P in the plane with respect to OX.

If we draw PN perpendicular to OX, then with respect to axes through O along OX and perpendicular to OX

$$ON = x \text{ and } PN = y,$$

and we have $\mathbf{x = r \cos \theta, \ y = r \sin \theta}$ **and** $\mathbf{r^2 = x^2 + y^2}$.

We shall use these relations to convert polar to Cartesian co-ordinates, the equation in Cartesian co-ordinates being in general a more convenient one to use for recognizing the degree, symmetry, etc., of the curve.

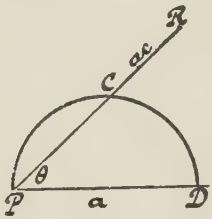


FIG. 59

To obtain the polar equations of the limaçon and cardioid, let PD be the diameter of the circle and its length a. Let PR $= r$, CR $= ac$ and $\angle$ CPD $= \theta$.

Then, since PCD is a right angle,

$$PC = a \cos \theta;$$

$$\therefore \quad r = PR = a \cos \theta + ac = a(c + \cos \theta) \quad \ldots \ldots \ldots \text{(I)}$$

This is the polar equation of the limaçon, and it becomes the equation of the cardioid when $c = 1$

To convert to Cartesian co-ordinates multiply by r,

$$r^2 = acr + ar \cos \theta;$$

and, using the substitutions given above,

$$x^2 + y^2 = acr + ax,$$

i.e. $$x^2 + y^2 - ax = acr;$$

square, $$(x^2 + y^2 - ax)^2 = a^2c^2r^2 = a^2c^2(x^2 + y^2) \ldots \text{(II)}$$

But these curves are also connected with the Conic Sections, and we can see this connexion most simply by using the polar equation.

Polar Equation of a Conic.—Let S be the focus, XK the directrix of a conic, e its eccentricity, Q a point on it.

Let SX $= d$.

Then if SQ $= r$, since SQ $= e$QK (property of a conic);

$$\therefore \quad QK = \frac{r}{e}$$

and $$d = SX = SN + NX = r \cos \theta + \frac{r}{e};$$

i.e. $$r(1 + e \cos \theta) = ed.$$

This can be transformed to a second degree equation in Cartesian co-ordinates.

Now let P be a point on SQ, such that SQ . SP $= b^2$.

Then $$\text{SP} = \frac{b^2}{\text{SQ}} = \frac{b^2(1 + e\cos\theta)}{de};$$

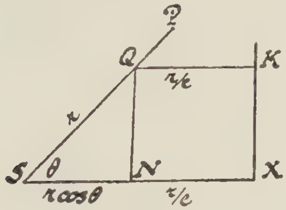


FIG. 60

and substituting a for b^2/d and $1/c$ for e we reduce this to the form of equation (I), i.e. the locus of P is a limaçon or a cardioid.

If $e > 1$, it is the limaçon with a loop,
if $e < 1$, it is the limaçon without a loop,
if $e = 1$, it is the cardioid.

Inversion.—If Q is a point on one curve, and if along a line SQ, drawn from a fixed point S to Q, R is taken, such that SR . SQ is constant, then the locus of R is said to be the inverse of the locus of Q (and vice versa) w.r.t. (i.e. with respect to) S.

We see, then, that the limaçons are inverses of the hyperbola and ellipse, and the cardioid is the inverse of the parabola w.r.t. a focus.

Plotting Inverse Curves.—Draw a branch of the rectangular hyperbola $xy = b^2$, Fig. 61 (1). If P is any point on it and PN and ON are the ordinate and abscissa, then ON . PN $= b^2$.

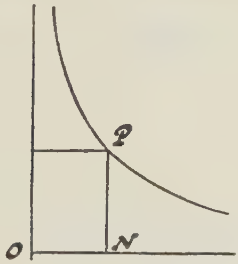


FIG. 61 (1)

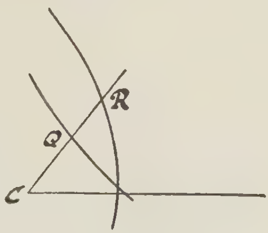


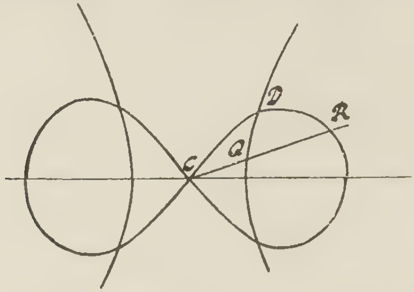
FIG. 61 (2)

To obtain the inverse of any curve with respect to a fixed point C, let Q be any point on the curve, Fig. 61 (2).

Take an abscissa of the hyperbola ON equal to CQ, then the corresponding ordinate PN gives the length CR.

By plotting in this way the inverse curve can be traced.

Booth's Lemniscates.—Using the above method, plot the inverse of the ellipse and hyperbola w.r.t. the centre.



Booth's Hyperbolic Lemniscate

FIG. 62

The curves obtained are called Booth's hyperbolic and elliptic lemniscates (Gk. *lemniskos* = *ribbon*).

Equation of Hyperbolic Lemniscate.—Let Q be a point (x, y) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Transforming to polar co-ordinates,

$$\frac{r^2 \cos^2 \theta}{a^2} - \frac{r^2 \sin^2 \theta}{b^2} = 1$$

$$r^2(b^2 \cos^2 \theta - a^2 \sin^2 \theta) = a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let OR be ρ and let $\rho r = d^2$, where d = CD.

Multiplying (1) by ρ^4

$$r^2\rho^2\rho^2 (b^2 \cos^2 \theta - a^2 \sin^2 \theta) = \rho^4a^2b^2,$$

i.e. $$d^4\rho^2(b^2 \cos^2 \theta - a^2 \sin^2 \theta) = \rho^4a^2b^2,$$

i.e. $$d^4(b^2\rho^2 \cos^2 \theta - a^2\rho^2 \sin^2 \theta) = (\rho^2)^2a^2b^2.$$

But the x, y for R are such that $\rho \cos \theta = x$, $\rho \sin \theta = y$, and $\rho^2 = x^2 + y^2$;

$$\therefore \quad d^4(b^2x^2 - a^2y^2) = (x^2 + y^2)^2a^2b^2.$$

If the hyperbola is a rectangular hyperbola, the lemniscate obtained is the Lemniscate of Bernouilli; in this case $a = b$, and the equation reduces to the form given on p. 90.

The resemblance between the equations of lemniscates and limaçons in degree and symmetry should be noticed.

The inverse of a circle may be a straight line, and this fact is used in the construction of a machine to draw a straight line.

Peaucellier's Cell. The Straight-line Machine.—A linkage of exceptional interest is the straight-line machine, or Peaucellier's Cell. A circle, theoretically correct, can be drawn by a pair of compasses; a straight line is drawn by the use of a ruler, but

although a line so drawn is regarded as straight for practical purposes, it has no theoretical justification. Peaucellier's cell, if perfectly constructed, would draw a straight line theoretically correct; in practice, it is difficult to make the linkage so that the line it draws is not obviously, to the eye, curvilinear.

Seven rods are needed, AB, BC, CD, DA, of equal length, pivoted at A, B, C, and D, to form a rhombus; BF and DF, equal in length, pivoted at B, D and F; CO pivoted at C.

O and F are fixed, and the distance OF = OC.

A traces a straight line.

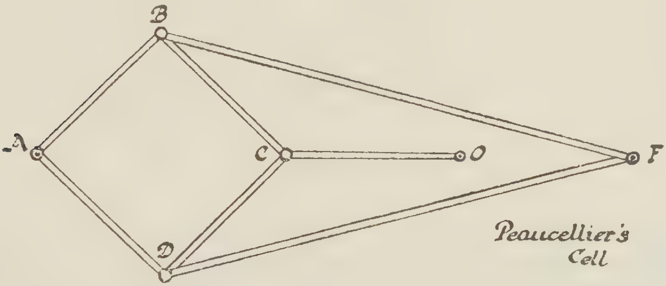


FIG. 63

In order that BF and DF may clear C, they may be made concave to the plane; the shape of the rods does not matter, provided the lengths between the pivots are correct.

In any position of the linkage the figure formed by the five points A, B, C, D, and F is symmetrical about ACF, which is a straight line, and C and F are on the circumference of a circle of centre O.

In Fig. 64 let ABCDF be any position of the linkage. Since ABCD is a rhombus, AC and BD bisect at right angles at G; and the rectangle FC . FA

$$\begin{aligned} FC \cdot FA &= (FG + GA)(FG - GC) \\ &= (FG + GA)(FG - GA) \\ &= FG^2 - GA^2 \\ &= (FB^2 - BG^2) - (BA^2 - BG^2) \\ &= FB^2 - BA^2, \end{aligned}$$

and is a constant.

Let FO meet the circumference of the locus circle of C in L, and let M be the position occupied by the tracing point A, when the pivot C is at L.

Then L is a fixed point, and since FL . FM is a constant, M is a fixed point.

Also since $FL \cdot FM = FB^2 - FA^2 = FC \cdot FA$

$\therefore$ the figure AMLC is cyclic and $\angle LMA = 180° - ACL = LCF = 90°$.

∴ A is always on a straight line perpendicular to FO through a fixed point M.

Since, as we proved, FC . FA = a constant, the locus of A is the inverse of the locus of C (and vice versa) w.r.t. F. And the

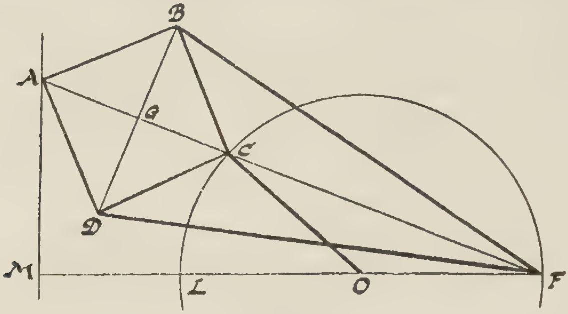


Fig. 64

linkage, without the rod OC, is an inversion machine. If this linkage could be used so that C is constrained to describe any curve, then A would describe the inverse.

Spirals.—The two Spirals—the **Spiral of Archimedes** and the **Equiangular Spiral**—have equations which are most simply expressed in polar co-ordinates.

To construct the spirals, draw a number of radial lines OA, OB, OC, OD, etc., at equal angular distances (say 30°). Along

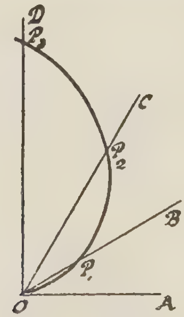


Fig. 65

them, commencing with OB, mark off OP_1, OP_2, OP_3, etc., so that these lengths are in arithmetical progression—say 1, 2, 3, . . . units. Then the points P are on the Spiral of Archimedes, whose equation is easily seen to be $r = c.\theta$, where c depends on the units of angle and length chosen.

(Paper ruled with circular and radial lines can be obtained for use in plotting.)

If P_1, P_2, P_3 are taken so that OP_1, OP_2, OP_3 . . . are in geometrical progression, say $1{\cdot}2, 1{\cdot}2^2, 1{\cdot}2^3$, . . . units, the points P are on the equiangular spiral, whose equation is of the form $r = c \,.\, b^\theta$ where c and b depend on the units of angle and length chosen.

The lengths OP_1, OP_2, OP_3 . . . can be constructed thus:

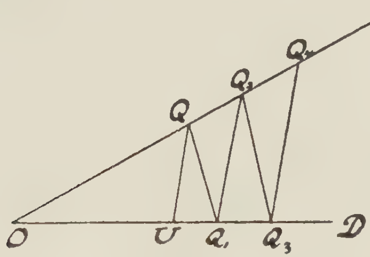


FIG. 66

Take any two lines OD and OE. From OD cut off a unit length OU. From OE and OD cut off OQ and OQ_1 each of the required length OP_1.

Draw Q_1Q_2 parallel to UQ, Q_2Q_3 parallel to QQ_1, Q_3Q_4 parallel to Q_1Q_2, and so on; then OQ_1, OQ_2, OQ_3, . . . are the lengths OP_1, OP_2, OP_3, . . .

Any curve which can be represented by an equation in which r is expressed as a function of θ can be plotted in the same way as the spirals.

For example, $r = a \sin n\theta$ is a rose-shaped curve with n or $2n$

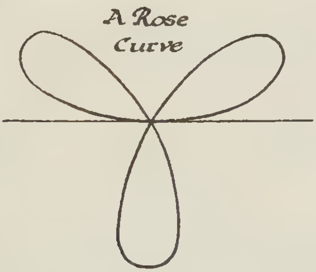


FIG. 67

petals, according as n is odd or even. The reader is advised to see why there is this difference.

The figure shows $r = a \sin 3\theta$.

The Curves, $r^n = a^n \cos n\theta$.—By giving to n different values, positive or negative, integral or fractional, we obtain a number of curves quite easy to plot and of considerable variety of form.

The following special cases are connected with curves dealt with in these chapters.

	Equation	Curve
$n = 1$	$r = a \cos \theta$; or $x^2 + y^2 = ax$.	circle.
$n = -1$	$a = r \cos \theta$; or $a = x$.	straight line
$n = 2$	$r^2 = a^2 \cos 2\theta$ or $(x^2 + y^2)^2 = a^2(x^2 - y^2)$	Bernouilli's lemniscate
$n = -2$	$a^2 = r^2 \cos 2\theta$ or $a^2 = x^2 - y^2$	rectangular hyperbola
$n = \frac{1}{2}$	$r = \frac{a}{2}(1 + \cos \theta)$	cardioid
$n = -\frac{1}{2}$	$a = \frac{r}{2}(1 + \cos \theta)$ or $y^2 = 4a(a - x)$	parabola

And it will be noticed that to change the sign of n is to obtain the equation of the inverse curve.

The Sine Graph.—The sine graph is a periodic undulatory curve of great importance in the Theory of Harmonic Motion and in all Wave Theory.

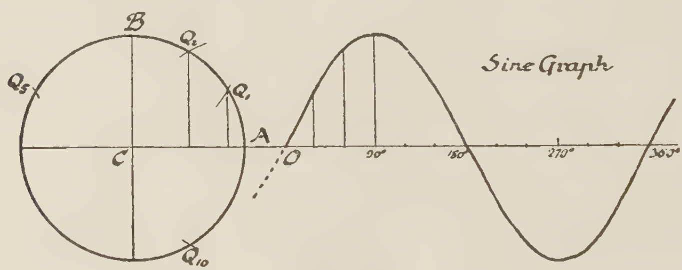


FIG. 68

It is plotted as follows:

Take a circle of unit radius, centre C, and let CA and CB be two radii at right angles to each other. With centres A and B and unit radii draw circles cutting the first circle in Q_1, Q_2, Q_5, Q_{10}; diameters through these points give other points, so that the circumference is divided into 12 arcs, each subtending 30° at the centre. On the x-axis mark off unit lengths to represent

30°, and for values of y take the lengths of the perpendiculars from the points Q to CA. The sine curve is obtained.

If the times of sunrise are plotted, the graph looks like a sine-curve, and by the exercise of a little ingenuity in the choice of axes and units the sunrise graph and the sine graph can be plotted so as to make the two very nearly coincide, i.e. the time of sunrise is approximately a sine function of the time of year. The graph of any function of the form $a \sin (n\theta + \alpha)$ or $a \cos (n\theta + \beta)$ has the same appearance.

Two or more such functions can be added to produce a periodic function whose graph is more complicated.

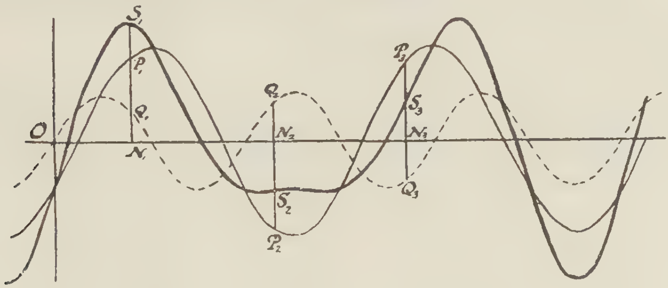


FIG. 69

Draw the graph of $\sin 3\theta$, OQ_1Q_2, . . . and of $2 \sin (2\theta - 30°)$, $P_1P_2P_3$ . . .

Take ordinates PQN and in them find S, so that SN = the algebraical sum of PN and QN. The curve $S_1S_2S_3$ is obtained. Its period is the L.C.M. of the periods of the two components.

It is possible conversely to resolve any periodic undulatory graph into a number of sine graphs. This is done in practice for the prediction of tides, and Lord Kelvin devised a machine to effect the resolution.

Lissajous' Figures.—The student of sound becomes acquainted with some graceful and complicated curves known as Lissajous' figures. The compound sine curves are connected with the combination of vibrations in the same direction; Lissajous' figures are connected with the combination of two vibrations not in the same direction but at right angles to each other.

These curves can be plotted thus:

Draw two circles of any radii. Take in each two diameters at right angles, so that one pair is parallel to the other. Divide the quadrants into equal parts, say 3 in one circle, 2 in the other. Through the points on the circumferences draw lines parallel to the diameters originally taken. A network of lines forming a number of rectangles is obtained. Starting at any point of intersection of this network, and proceeding to the diagonally opposite point of a small rectangle, draw a smooth curve. By varying the

number of divisions of the quadrants and by varying the starting-point for drawing the curve, a great variety of forms can be obtained. They can all be constructed by a mechanism consisting of two pendulums. One pendulum carrying a pen is swung so that the

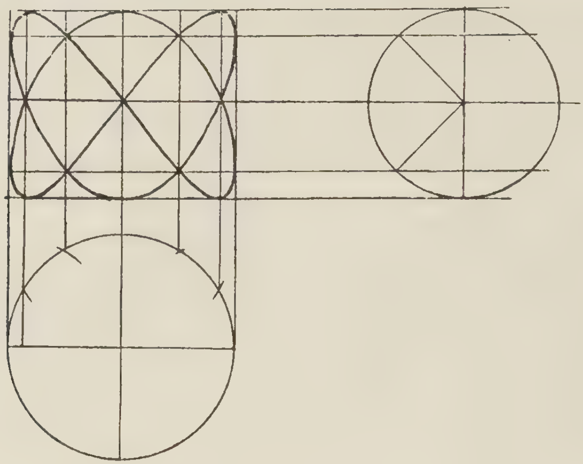


Fig. 70

pen is in contact with a board carried by the other pendulum, which is swung in a different direction. The times of swing of the pendulums are regulated so as to have the same ratio as the number of divisions used in the circumferences of the two circles in the graphic method. The figures are connected with the theory of notes in harmony.

Roulettes.—When one curve rolls without slipping so as to remain in contact with another, any point in the plane of the first curve, moving with it, traces out a roulette.

We shall confine ourselves to those cases in which the first curve is a circle and the second either a straight line or a circle.

The mechanical construction of these curves is very simple, and obvious enough. Geared wheels can be used to prevent slipping, but are not necessary. If the eye could follow the track of a clearly marked point on a cart wheel travelling along a level road, or on a wheel geared outside or inside a fixed wheel, it would see the shapes of these curves. It does not require much imagination to picture them.

The Cycloid.—The cycloid is traced by a point on the circumference of a circle which rolls in contact with a straight line. Its properties attracted the attention of Galileo, Pascal, Wren, and other seventeenth-century mathematicians.

Attach firmly a pencil point to the rim of a circular disc ;

lay a straight edge on paper and roll the circle along the straight edge; the pencil traces out a curve APDB, as in Fig. 71, with cusps at the points A and B where the circle meets the ruler.

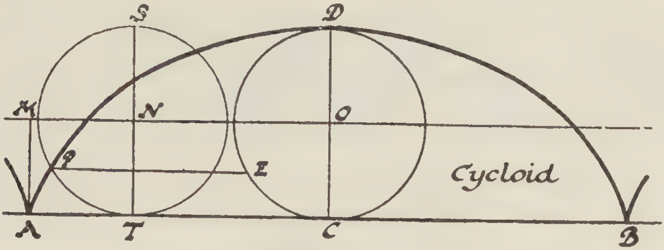


FIG. 71

This is the **cycloid** (Gk. *kuklos, a circle*). If the pencil were attached to positions within the circle of the disc or on a projecting flange the cycloid would be modified to the **trochoid** (Gk. *trochos, a wheel*) or **subtrochoid** forms, shown in Fig. 72.

FIG. 72

The three variations of form are comparable with the three forms of conchoids or the group of limaçons and cardioid.

Now, in Fig. 71, since the rolling circle has made one revolution between the cusps A and B, the straight line AB is $2\pi r$ in length.

Let C be the mid-point of AB, then AC = CB = πr; and let CD be the diameter of the rolling circle and O its centre, in the position CED when it has made half a revolution, then D is the position of the tracing-point and CD = $2r$.

Let SPT be any position of the rolling circle, T the point of contact, ST a diameter, and P the position of the tracing-point.

Draw PE parallel to AC to meet the near semi-circumference of circle CED in E.

Then, by the principle of rolling without slipping, the arc PT = AT, and also, since SPT = semi-circumference = ATC, $\therefore$ arc SP = TC = PE. This result will be referred to below.

Plotting the Cycloid.—Draw AM and TN perpendicular to AB to meet in M and N the path of the centre of the rolling circle, viz., the line through O parallel to AB, Fig. 71.

Then AT = arc PT and AC = semi-circumference; $\therefore$ whatever fraction the arc PT is of SPT or the arc EC is of CED, AT is the same fraction of AC, i.e. MN is the same fraction of MO. This enables us to plot the cycloid.

Take two lines at right angles, MO and COD as in Fig. 71. Let OC and OD be each one unit in length, and MO π units in length.

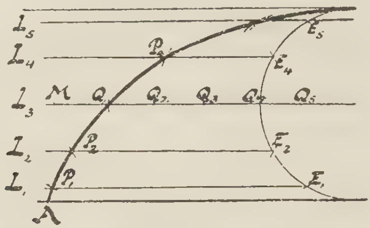


FIG. 73

Draw on CD as diameter a semicircle and divide the semi-circumference into any number, say 6, equal parts at E_1, E_2, E_3, . . . Through E_1, E_2, E_3, . . . draw lines L_1, L_2, L_3, etc., parallel to MO. Divide MO into the same number of equal parts at Q_1, Q_2, Q_3, . . . With centres Q_1, Q_2, Q_3 . . . and unit radii draw arcs to cut L_1, L_2, L_3, . . . in P_1, P_2, P_3, . . . Then the points P are on the cycloid.

The Area of the Cycloid.—Carry out this construction on squared paper, and, by counting squares or otherwise, find the area enclosed between the cycloid and AB, and compare it with the area of the circle. It will be found to be approximately $3\pi r^2$. It can be proved to be $3\pi r^2$.

Find in the same way the area of the claw-shaped figure enclosed between the semi-cycloid APD, the semi-circumference CED and the line AC. It will be found to be approximately πr^2; it can be proved to be πr^2.

Take two positions of the rolling circle, giving points P and p

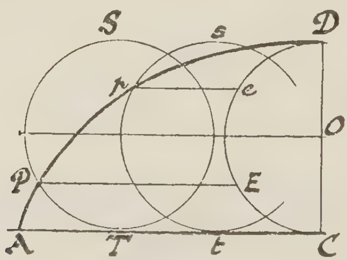


FIG. 74

on the cycloid such that P is as far above AC as p is below D. Let lines through P and p parallel to AC meet CED in E and e. Let TS and ts be the diameters of the circles through the points of contact.

Then, using the relation established above,

$$AT = \text{arc } PT = \text{arc } EC = \text{arc } eD = \text{arc } sp = pe,$$
$$\text{and } TC = PE$$
$$\therefore \quad pe + PE = AC = \pi r.$$

Suppose the claw area to be divided into a large number ($2n$) of strips of the same width h, by lines parallel to AC. $2n$ can be so large that the strips are approximately rectangles, and if a pair be taken with PE and pe as bases, their combined area is $(PE + pe)h$, i.e. πrh. Adding all similar pairs, we get the whole claw to be of the area $\pi r.nh$, i.e. πr^2.

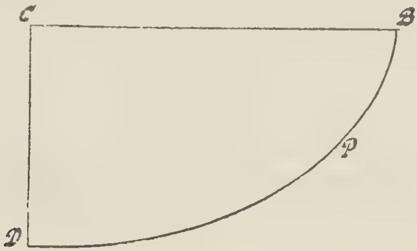


FIG. 75

Dynamical Property of the Cycloid.—An inverted cycloidal groove with the cusps in a horizontal straight line has this property: that if a particle slide or roll down from a point P it will arrive at the lowest point D in the same time, wherever the point P is taken (Fig. 75).

Epicycloid and Hypocycloid.—If the rolling circle roll in contact with another circle instead of a straight line, a point on its circumference traces out an epicycloid (Gk. *epi*, *on*) or hypo-

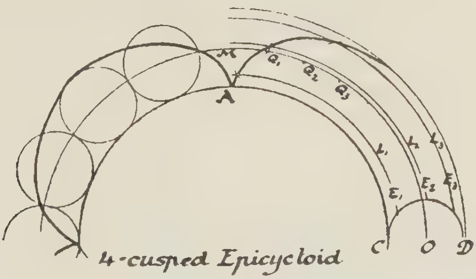


FIG. 76 (1)

cycloid (Gk. *hupo*, *under*), according as it rolls outside or inside. The figures show a four-cusped epicycloid and a four-cusped hypocycloid. They are lettered in the same way as the cycloid, Fig. 73, so that the geometrical construction can be carried out by

following the same directions with only such slight alterations as will be obvious.

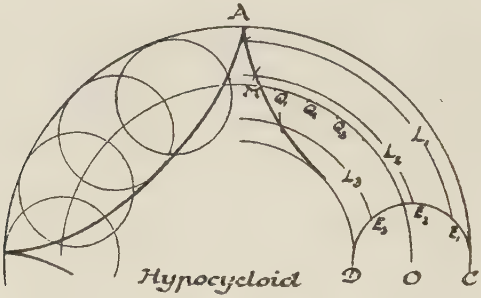


FIG. 76 (2)

The One-cusped Epicycloid.—Let C be the centre of a fixed circle, O the centre of the rolling circle of the same radius.

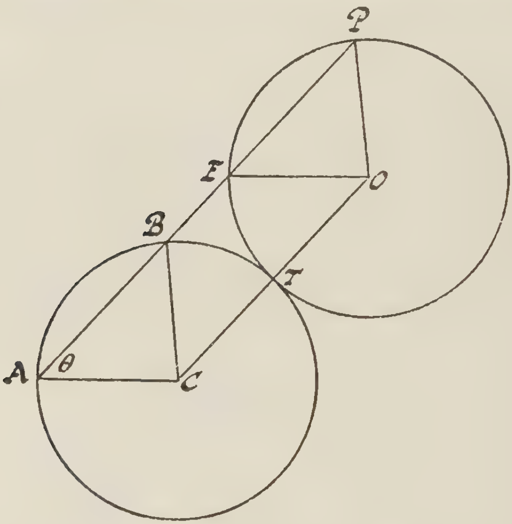


FIG. 77

There will be one cusp; let it be at A, and let P be another position of the tracing-point.

Join AP, cutting the circles in B and E. Join CB, EO, and CO.

It is then easily proved that BCOP and ACOE are parallelograms. Then BP = CO = the diameter of each circle, and hence the locus of P is a cardioid.

This suggests another mechanical means of plotting the cardioid.

Make a parallelogram linkage ACOE, such that AE = 2AC, and having AE prolonged with a slot along its extension. Attach a rod OP of the same length as AC, so that it is pivoted at O, and a tracing-point P can move along the slot. With AC fixed, P traces out a cardioid.

The Two-cusped Hypocycloid.—Consider a circle rolling inside a circle of centre O and of twice its diameter. O is always on the circumference of the rolling circle.

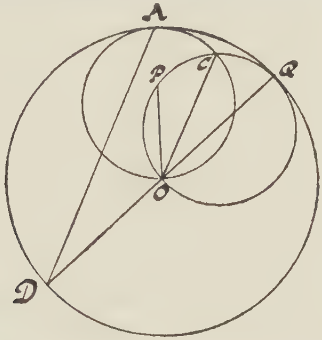


FIG. 78

Let A be a point where the tracing-point is in contact with the outer circle, P the position of the tracing-point when the circles are in contact at Q, and let C be the mid-point of the arc PQ.

Then, by the property of rolling, arc PQ = arc AQ; ∴ arc CQ = half arc AQ. But the radius of the arc CQ = half the radius of the arc AQ; ∴ these arcs subtend equal angles at the circumferences of their respective circles;

∴ $\angle COQ = \angle ADQ$;

∴ $\angle AOQ = 2 \angle ADQ = 2 \angle COQ = \angle POQ$;

∴ APO is a straight line; i.e. P traces out a straight line, and the two-cusped hypocycloid is a diameter of the fixed circle.

This suggests another mechanical construction for drawing a straight line.

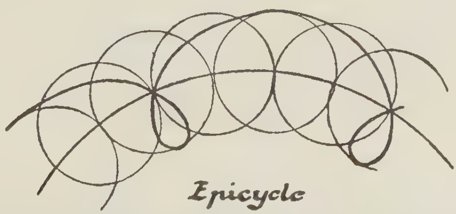


FIG. 79

Epicycles.—The epicycloid was suggested by the epicycle. The latter is traced out by a point on the circumference of a circle

which rotates uniformly about its centre, while its centre traces out with uniform speed the circumference of another circle.

Astronomers previous to Kepler used epicycles as approximations to the orbits of the planets.

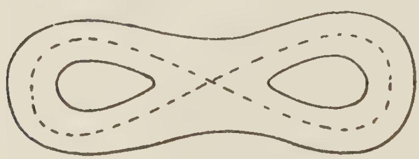
Plotting from Two Fixed Points of Reference.—The position of a point in a plane is fixed in Cartesian co-ordinates by its distances from two fixed lines ; in polar co-ordinates by one distance (from a point) and an angle ; it can also be fixed by its distances from two fixed points.

FIG. 80

Let H and S be two fixed points and P a variable point. Let PS and PH be r_1 and r_2, then a law connecting r_1 and r_2 will determine the locus of P.

The following can be readily plotted in this way :

$r_1 = r_2$, the right bisector of HS.
$r_1^2 - r_2^2 = d^2$, a line perpendicular to HS.
$r_1 / r_2 = c$, a circle.
$r_1 + r_2 = 2a$, an ellipse with H and S as foci.
$r_1 \sim r_2 = 2a$, a hyperbola with H and S as foci.
$r_1 r_2 = d^2$, one of Cassini's ovals, of which Bernouilli's Lemniscate is a special case, if $2d$ is HS.
$r_1 + lr_2 = d$, Oval of Descartes ; and of this $r_1 = r_2$, $r_1 + r_2 = 2a$, $r_1 \sim r_2 = 2a$ are special cases.



Cassini's Ovals

FIG. 81

There are other curves to be plotted by simple means. It has probably occurred to the reader that by taking the point P inside or outside the circle, Fig. 55, and using the limaçon construction, variations would be obtained, and that there are any number of

curves to be obtained by inversion. In those we have dealt with he has probably noticed that there is inter-connexion among many of them: the circle is a conic section, the limaçons and cardioid are connected both with the circle and the other conics, the lemniscates are connected with the conics.

To carry this connexion a step farther we might consider some cases of a family of curves that can be plotted, though not by the simple ways mentioned in this chapter.

Pedal Curves.—If T is a point on a curve, TP the tangent at T, OP a $\perp$ to TP, then the locus of P is the pedal of the locus of T.

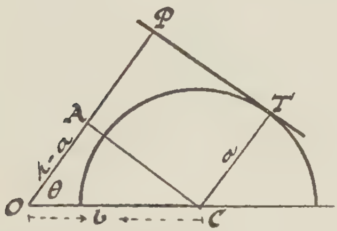


FIG. 82 (1)

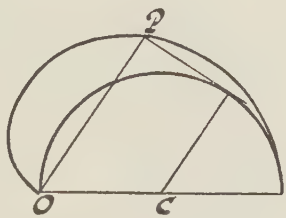


FIG. 82 (2)

Consider a circle of radius a, centre C; O, a fixed point. From O draw a $\perp$ OP to the tangent at T.

Let CA be perpendicular to OP.

Then, since $\angle$ OAC is a right angle, the locus of A is a circle on OC as diameter. The chord OA is produced so that AP = CT = a constant, i.e. the locus of P is a limaçon or cardioid. Part of the cardioid is shown in Fig. 82 (2).

With a linkage contrived to keep (1) A on the circumference of a circle of which OC is the diameter, and (2) CAPT a parallelogram, P would trace out these curves.

A rod OP pivoted at O and connected so as to be always parallel to CT of a $\llcorner$-piece CTP pivoted at C could be arranged to make P trace out the curves.

Pedals of any curve w.r.t. any point can be plotted and discussed. Those of the conic sections w.r.t. the centre or focus are the most worthy of attention.

CHAPTER VI

PYTHAGORAS' THEOREM

THE right-angled triangle holds a position of special importance in mensuration. Any triangle can be regarded as the sum of two right-angled $\triangle$s ; an obtuse-angled $\triangle$ may also be regarded as the difference of two right-angled $\triangle$s. Hence, if we can show that the angle-sum of a right angled $\triangle$ is 2 right $\angle$s, we can show that the angle-sum of any $\triangle$ is 2 right $\angle$s ; if we can show that the area of a right-angled $\triangle$ is half the product of the base and altitude (the hypotenuse not being the base), we can prove the same formula for any triangle.

Pythagoras' Theorem gives a specially important relation connecting the lengths of the sides of a right angled $\triangle$; and this is extended in Euclid's geometry to give a relation which is equivalent to the important formula $c^2 = a^2 + b^2 - 2ab \cos C$.

Pythagoras.—Pythagoras was a Tyrian, born at Samos, about 570 B.C. He studied under Anaximander, a pupil of Thales, the father of Geometry. After travelling in Egypt and Asia Minor he started to lecture at Samos. Meeting with little success, he migrated with his mother and one disciple to the Greek settlements in South Italy known collectively as Magna Græcia. At Croton his lectures were crowded with enthusiastic pupils, and he became the head of a large and influential society.

The majority of his disciples were probationers ; to them were communicated the results of discoveries, the formulas of his philosophy. An inner circle of esoterics formed a group of fellow-workers and investigators, who rapidly extended the bounds of mathematical knowledge. So great was Pythagoras' influence, and so enthusiastic the devotion of his school, that whatever they discovered was ascribed to The Master under the formula ἀυτὸς ἔφη, a formula for authoritative utterance that has come to us in the Latin translation, "ipse dixit." Hippasus, who discovered the regular dodecagon, was drowned by his fellow-disciples for taking to himself credit for the discovery.

The school lived an abstemious life as a disciplined community ; but their influence aroused suspicion and at last open hostility. Pythagoras was killed in a riot fomented by political opponents. But the Pythagoreans maintained themselves for a hundred years in Tarentum as a Mathematical and Philosophical School ;

their doctrines and discoveries spread and formed the framework of Greek mathematics.

The deductive method of Greek geometry is due to Pythagoras, whose boast it was that "he raised arithmetic above the needs of the merchant."

His Work.—He is credited with most of the geometry of triangles, parallelograms, and rectangles.

He proved that the plane space about a point can be entirely filled with (1) equilateral triangles, (2) squares, (3) regular hexagons.

His school knew that there were five regular solids inscribable in a sphere.

They were probably acquainted with the Theory of Proportion, and they certainly knew something of irrational numbers.

In arithmetic they investigated properties of numbers.

Their Arithmetical Theory was based not on algebraic symbols but geometrical illustration. Thus, a number was represented by the length of a line, a very convenient mode of dealing with irrationals derived from square roots. He summed the series $1 + 3 + 5 + \dots$ to n terms by means of a dissected square (*see* p. 264).

Interwoven with their mathematical work was a certain amount of mystical philosophy: perfection was to be found in the circle, the sphere, the number 4; the causes of colour resided in the properties of the number 5; the explanation of fire was to be sought in the nature of a pyramid.

Metempsychosis or the belief in the transmigration of the soul was their chief non-mathematical tenet.

The Theorem.—Egyptians and Chinese at an early date knew that, if the sides of a triangle measure 3, 4, and 5 units, the triangle is right-angled. The Egyptians used this knowledge in laying out the bases of the pyramids. "Rope-stretchers" were employed.

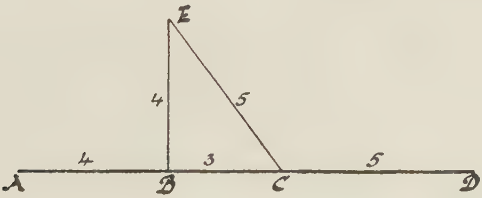


FIG. 83

In Fig. 83, ABCD is a rope, AB of 4, BC of 3, CD of 5 units of length. BC is placed along an oriented base line with B at the point where the right angle is to be. The rope-stretchers take the ends A and D and move till they meet at some place E with the ropes taut; $\angle$ EBC is then a right angle.

A man with his mind bent on the consideration of area, contemplating a floor of square tiles with diagonal markings would

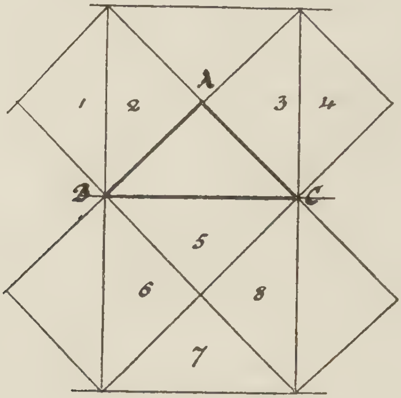


FIG. 84

see that for the right-angled $\triangle$ ABC (Fig. 84) the square on AB + the square on AC ($\triangle$s 1 and 2 + $\triangle$s 3 and 4) = square on BC ($\triangle$s 5 + 6 + 7 + 8).

If he found that $5^2 = 4^2 + 3^2$, he would see that the same enunciation $AB^2 + BC^2 = AC^2$ holds for the triangle of the rope-stretchers. He might inquire if this relation of squares was generally true. Pythagoras proved that it was. His method of proof is conjectured to be one (perhaps both) of the following :

Proof 1.—Let ABC be the triangle, C being the right angle, a, b, c the sides. Complete the square on $a + b$. Dissect it in two ways, as in Fig. 85 (1) and Fig. 85 (2).

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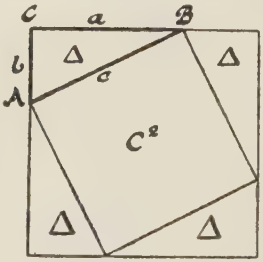


FIG. 85 (1)

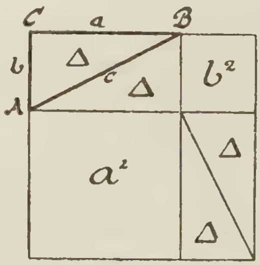


FIG. 85 (2)

In Fig. (1), $(a + b)^2 = c^2 + 4\triangle$,
in Fig. (2), $(a + b)^2 = a^2 + b^2 + 4\triangle$;
whence $c^2 = a^2 + b^2$

Proof 2.—Take the $\triangle$ ABC as before with C the right angle. Draw CD perpendicular to AB. Then by similar triangles,

$$\frac{AD}{AC} = \frac{AC}{AB}, \text{ i.e. } AC^2 = AD \, . \, AB \quad \dots\dots\dots \quad (1)$$

$$\frac{BD}{BC} = \frac{BC}{AB}, \text{ i.e. } BC^2 = DB \, . \, AB \quad \dots\dots\dots \quad (2)$$

Adding (1) and (2), $AC^2 + BC^2 = AB^2$.

The steps (1) and (2) of this proof are steps in the usual proof,

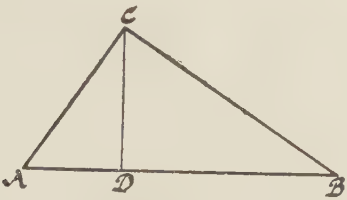


FIG. 86

which is due to Euclid. But Euclid is compelled to dispense with similar triangles for reasons of sequence. The figure of Euclid's proof is known by the French as *pons asinorum*, by the Arabs as the "Figure of the Bride."

An American scientific publication collected over a hundred different proofs. Some of these are only slight variants of others; many are dissection proofs, and some prove steps (1) and (2) of Proof 2 by other methods.

We give some proofs.

Proof 3.—BHASKARA (A.D. 1114), draws this figure and says, "Look and See."

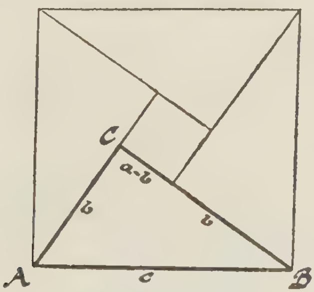


FIG. 87

The explanation, which he does not give, is:

$$c^2 = 4 \triangle ABC + (a-b)^2$$
$$= 2ab + (a-b)^2 = a^2 + b^2.$$

Compare this with Proof 1.

Proof 4.—On BC describe the square BCDF, make FE in FD of length b.

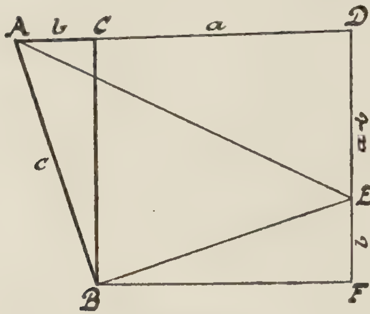


FIG. 88

Then $$a^2 = \text{CF} = \text{BEF} + \text{BEDC}$$
$$= \text{ABC} + \text{BEDC}$$
$$= \text{BAE} + \text{ADE}$$
$$= \tfrac{1}{2}c^2 + \tfrac{1}{2}(a + b)(a - b)$$
$$= \tfrac{1}{2}c^2 + \tfrac{1}{2}a^2 - \tfrac{1}{2}b^2;$$
whence $$c^2 = a^2 + b^2.$$

Proof 5.—This is due to the late C. S. JACKSON ("Slide

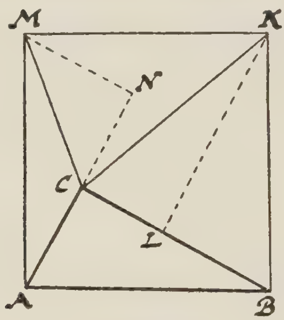


FIG. 89

Rule Jackson"): Complete the square ABKM. Draw KL and MN perpendicular to BC and AC. Join CK and MC;

$$\text{MN} = \text{BL} = b,$$
$$\text{AN} = \text{LK} = a,$$

$\triangle$s MCA and BCK together make $\frac{1}{2}$ ABKM;

$$\therefore \quad \tfrac{1}{2}b^2 + \tfrac{1}{2}a^2 = \tfrac{1}{2}c^2.$$

Proof 6.—This is the simplest and best known of the dissection proofs. It will be seen to be adapted from Proof I.

Construct the figure made up of $a^2 + b^2$, as in Fig. 90 (I).

Take away the shaded parts and replace them as in Fig. 90 (2). The square c^2 is formed.

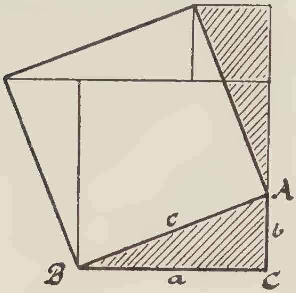


FIG. 90 (1)

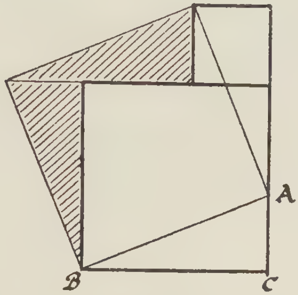


FIG. 90 (2)

Proof 7.—O is the intersection of the diagonals of the square on CB. LOM and HON are perpendicular and parallel to AB, and each $= c$.

Draw OZ perpendicular to CE.

Then $\triangle$ HOZ is similar, and similarly placed, to ABC, and its sides are half the length of the sides of ABC.

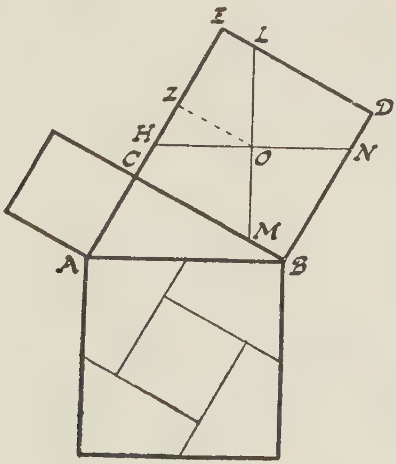


FIG. 91

Hence it can be proved that CH $= \frac{1}{2}(a - b)$, HE $= \frac{1}{2}(a + b)$, and the square on AC and the four parts of the square CBDE can be shown to fit together to form the square on AB.

There are many other dissection proofs; the merit of this is the idea of symmetry underlying it.

Proof 8.—Pappus' Theorem (*c.* A.D. 300). Let ABC be any triangle; and let AKLC and CMNB be any parallelograms described on AC and CB outwards (Fig. 92).

Let KL and NM meet in Q.

On AB describe a parallelogram AEFB, having AE equal and parallel to QC.

Let QC meet AB and EF in P and D, and let FB meet MN in O.

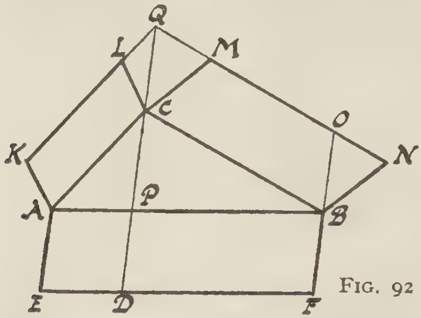


FIG. 92

Then since parallelograms on equal bases and between the same parallels are equal in area,

CN = QB = PF, and similarly KC = AD;

∴ sum of parallelograms on AC and CB = parallelogram on AB drawn as in this construction.

If C is a right angle and the parallelograms on AC and CB are squares, AF becomes the square on AB, and Pythagoras' Theorem is seen to be a particular case of Pappus' Theorem.

Proof 8a. Direct proof based on No. 8.—Draw the squares on the sides of the right-angled △ ABC (Fig. 93).

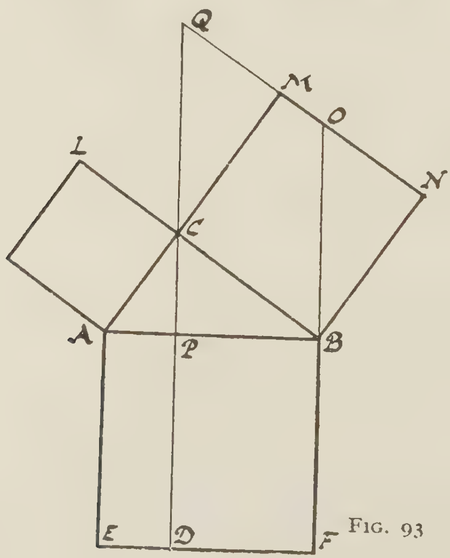


FIG. 93

Through C draw QCPD perpendicular to AB.
Produce FB to meet MN in O.
Then, as in Pappus,

$$CN = QB = PF.$$

This is step (1) of Proof 2. This step is recurrent in proofs of Pythagoras, and is otherwise important as giving a method of reducing a rectangle PF to a square of the same area CN.

CB is the geometric mean (or mean proportional) of AB and BP.

Proof 9. Leonardo da Vinci's Proof (1452–1519).—With figure and lettering as in Proof 8a, describe ERF on EF, so

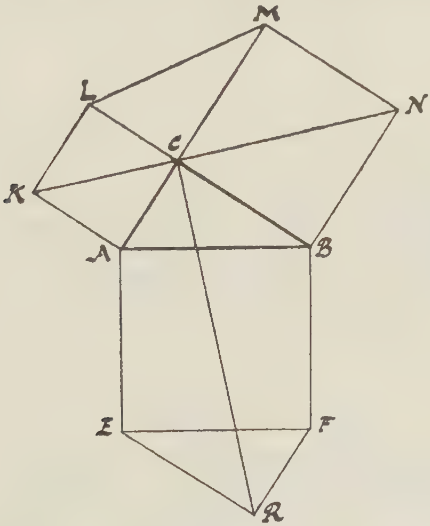


FIG. 94

that ER = BC, RF = AC. Then KCN is a straight line dividing ABNMLK into two equal quadrilaterals, each = ERCA = CRFB

$$\therefore \quad ABNMLK = AERFBA.$$

Take away two equal $\triangle$s and $a^2 + b^2$ remains equal to c^2.

Proof 10. Ptolemy's Theorem (A.D. 87–168).—If ADBC is any cyclic quadrilateral, AD . BC + AC . BD = AB . CD.

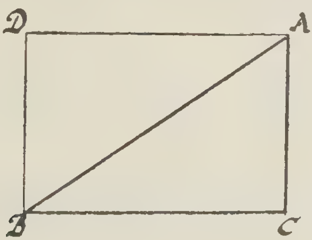


FIG. 95

If the quadrilateral is a rectangle whose sides are a and b and diagonal c, this gives the special case $a^2 + b^2 = c^2$.

Proof 11. Pythagoras' theorem is a special case of another proposition, viz. if AC bisects $\angle$ BAD of the $\triangle$ ABD, then

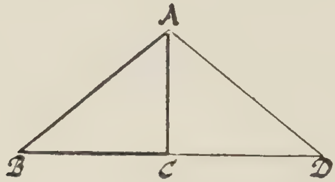


FIG. 96

AB . AD = AC² + BC . CD. If BA = AD, then AC is perpendicular to BD and $c^2 = b^2 + a^2$.

Incommensurability of $\sqrt{2}$.—In an isosceles right-angled triangle if the equal sides are a and a, the hypotenuse is $a\sqrt{2}$. Pythagoras proved $\sqrt{2}$ to be incommensurable with 1. As this is the first recorded investigation of incommensurables, we give his proof (it is included in Euclid's Bk. X).

He proves that the hypotenuse and a side are incommensurable (*see* p. 61).

Let c and a have no integral common factor. $c^2 = 2a^2$; $\therefore$ c is even. Let $c = 2p$.

Then $$4p^2 = 2a^2, \text{ i.e. } 2p^2 = a^2;$$

$\therefore$ a is even, which is impossible, since a is prime to c;

$\therefore$ c and a are incommensurable with each other.

Prime Rational Right-angled Triangles. Pythagoras' Formula.—Pythagoras knew that sides whose lengths are $2n^2 + 2n + 1$, $2n^2 + 2n$, $2n + 1$ form a right-angled triangle.

Clearly, by putting integral values for n we can find an unlimited number of right-angled triangles whose sides are integers and therefore rational. Tabulate some:

n	$2n^2 + 2n + 1$	$2n^2 + 2n$	$2n + 1$
1	5	4	3
2	13	12	5
3	25	24	7
4	41	40	9

The first is the 3, 4, 5 triangle of the rope-stretchers.

All triads of numbers a, b, c that would satisfy this formula can be obtained in this way:

Take any odd square, e.g., 11^2, i.e. 121. Divide it into two parts as nearly as possible equal, i.e. differing by 1. We get 61 and 60.

Then 61, 60, 11 is the triad.

For $61^2 - 60^2 = (61 + 60)(61 - 60) = 121 \times 1 = 11^2$.

Plato's Formula.—PLATO (*b.* 429 B.C.), according to Proclus, obtained the formula, $m^2 + 1, m^2 - 1, 2m$ for rational triads satisfying $c^2 = a^2 + b^2$.

Tabulate again:

m	$m^2 + 1$	$m^2 - 1$	$2m$
2	5	3	4
3	10	8	6
4	17	15	8
5	26	24	10
6	37	35	12

Again the first triad is 3, 4, 5. But here whenever m is odd we get a common factor for the members of the triad. Plato's 26, 24, 10 is the same as Pythagoras' 13, 12, 5. But Pythagoras does not get the triad 17, 15, 8, or 37, 35, 12. Nor does Plato get the triad 25, 24, 7, at any rate in this numerical form, but if $m = 7$ he gets 50, 48, 14.

General Formula.—Three data are necessary to determine a triangle; if one is given, e.g., that the triangle is right-angled, two others are required. We should expect, therefore, that in a formula for the sides there would be two unknowns. As neither Plato's nor Pythagoras' formula contains two unknowns, they cannot be regarded as covering all cases.

We shall find a formula that does, limiting ourselves to rational, integral triads which have no common factor, e.g., 5, 4, 3 will be regarded as the same as $\frac{5}{11}, \frac{4}{11}, \frac{3}{11}$, or 10, 8, 6.

With this restriction applied to Plato's triads we see that c (the measure of the hypotenuse) always differs from the measure of the next greater side by 2 or 1; in Pythagoras' triads c always differs from the next by 1. Plato's includes all Pythagoras', as will be seen if $2n + 1$ is substituted for m; but Pythagoras' will not include all Plato's; and such a case as 29, 21, 20 is not included in either.

Now, any number can be split into prime factors, which can be arranged as a product of the form m^2p, where m and p need not be prime and either may be unity and where m^2 includes all square factors; thus

$$3 = 1^2 . 3$$
$$4 = 2^2 . 1$$
$$120 = 4 . 30 = 2^2 . 30$$
$$180 = 36 . 5 = 6^2 . 5.$$

Again, if any other number n^2q multiplies m^2p to produce a perfect square, then $q = p$.

For $\qquad m^2p \,.\, n^2q$ is a perfect square ;

$\qquad \therefore\ pq$ is a perfect square.

But this is impossible, unless each factor of p is contained in q, and vice versa, i.e., unless $p = q$.

If c, a, and b are the measures of the sides of a right-angled triangle such that $c^2 = a^2 + b^2$,

Then
$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= (c + a)(c - a). \end{aligned}$$

Now, let $c + a = m^2p$; then, since $(c + a)(c - a)$ is a perfect square, $c - a$ is n^2p.

By addition $\qquad 2c = (m^2 + n^2)p$;

by subtraction $\qquad 2a = (m^2 - n^2)p$;

and by substitution for $c + a$ and $c - a$ above,

$$b = mnp.$$

We have, then,

$$c = (m^2 + n^2)\frac{p}{2},$$

$$a = (m^2 - n^2)\frac{p}{2},$$

$$b = 2mn\left(\frac{p}{2}\right);$$

rejecting the common factor $\frac{p}{2}$, we get the formula $\mathbf{m^2 + n^2, m^2 - n^2, 2mn}$ for the triad.

We shall find that if m and n have a common factor g, then the triad has a common factor g^2. And if m and n are both odd, the triad has a common factor 2. Tabulate, avoiding such cases.

m	n	$m^2 + n^2$	$m^2 - n^2$	$2mn$
2	1	5	3	4
3	2	13	5	12
4	1	17	15	8
4	3	25	7	24
5	2	29	21	20
5	4	41	9	40
6	1	37	35	12
6	5	61	11	60

Pythagoras' formula is a particular case of the general when $m = n + 1$. Plato's is obtained by putting $n = 1$.

Properties of Prime Rational Triads.—The first triad is 3, 4, 5. In the others it will be seen that one member of each is a multiple of 3, one a multiple of 4, and one a multiple of 5.

One Member is a Multiple of 3.—If either m or n is a multiple of 3, then $2mn$ is a multiple of 3.

If not, m and n are of the form $3p \pm 1$, $3q \pm 1$; m^2 and n^2 are of the form $9p^2 \pm 6p + 1$, $9q^2 \pm 6q + 1$; therefore $m^2 - n^2$ in any of these cases is a multiple of 3; $m^2 + n^2$ cannot be, i.e; one side, not the hypotenuse, is a multiple of 3.

One Member is a Multiple of 4.—m and n, as we have seen, cannot both be odd; one must be even. $2mn$ is a multiple of 4, i.e. one side, not the hypotenuse, is a multiple of 4.

One Member is a Multiple of 5.—If m or n is a multiple of 5, $2mn$ is a multiple of 5.

If not, m and n must be of the form $5p \pm 1$ or $5p \pm 2$ and $5q \pm 1$ or $5q \pm 2$; i.e. m^2 and n^2 are of the form $25p^2 \pm 10p + 1$ or $25p^2 \pm 20p + 4$, and $25q^2 \pm 10q + 1$, or $25q^2 \pm 20q + 4$.

And it will be seen that either $m^2 + n^2$ or $m^2 - n^2$ is, in any one of these cases, a multiple of 5; i.e. one side, which may be the hypotenuse, is a multiple of 5.

If either a or b is prime, the other differs from c by 1.

For
$$a^2 = c^2 - b^2$$
$$= (c + b)(c - b).$$

If a is prime, either $c + b = c - b = a$, which is impossible, or $c + b = a^2$ and $c - b = 1$.

Fermat's Last Theorem.—As $c^2 = a^2 + b^2$ has integral solutions, mathematicians have looked for integral values of a, b, and c to satisfy $c^3 = a^3 + b^3$, $c^4 = a^4 + b^4$, etc.

Fermat (1601–1665) stated that it is impossible to find integers c, a and b to satisfy $c^n = a^n + b^n$, if n is an integer > 2.

His proof for $n = 3$ is lost. Euler has supplied one.

Fermat's proof for $n = 4$ is extant.

To find a proof for all cases of n has occupied the attention of many mathematicians, and perhaps the prize offered for the solution has helped to produce greater keenness. There are some who claim to have solved the problem and won the prize, but their claims have not been accepted.

CHAPTER VII

SYMBOLS AND CONVENTIONS

A CHILD'S drawing of a man or a house bears sufficient resemblance to the real thing for it to be clear what the drawing represents. It is not however an exact picture; it is not a portrait, though perhaps it is intended to be. If, however, he deliberately puts on paper certain lines or colour-masses which recall the chief features of the object, intending that they shall serve to indicate but not to depict, and more especially if he intends the drawing to show certain relationships of the parts, then he is making a diagram or diagrammatic representation.

The botanist and zoologist use this method of representation and introduce arbitrary conventions of thickness of line, of colour, etc., which, being partly suggested by the character of the parts of the object, make a more immediate appeal to the eye and a more lasting impression on the mind. In the same way the cartographer and meteorologist use colour, hachures and contour-lines to represent certain characters of a piece of country and of weather conditions.

The mathematician, too, uses diagrams; sometimes they are as literal as the child's drawing of a house; his geometrical diagrams possess the essential characteristic of the concrete thing they represent; a triangle drawn on paper is probably not of the same shape and still less of the same size as some real triangle, but it has three sides and three angles and serves for reference in an investigation of the properties of all triangles. Sometimes his diagrams may be based on conventions, as an ordnance survey map is. The Greeks used lengths of lines to represent magnitudes, and developed a treatment of arithmetic in which lines and areas served the same purpose as some of the symbols in modern algebra. In the same way the modern mathematician by using lines to represent the metrical properties of force, velocity, etc., is able to discuss and develop the science of mechanics on the same lines as geometry. In graphical work, he uses diagrams in which the representation is conventional, e.g. abscissæ may represent time; ordinates, velocity; the gradients of tangents, accelerations; areas, distances, and so on.

But these are examples of highly developed modes of representation. The symbols of arithmetic and algebra, the numerals and the letters of the alphabet, are among the first means of communicating ideas that the child learns. In their origin

they were probably picture-diagrams, or ideograms. Just as the child will endeavour to draw a picture of a man long before he has any notion of the letter combination "MAN," so primitive man used a picture language before he had written symbols to represent sounds. From the picture language the alphabet was derived.

The characters of the Chinese language to this day are ideograms, being pictures conventionalized. The A of the Romans and the α of the Greeks are derived from a Phœnician diagram of an ox's head, which came into use to represent the first sound of the word for ox, and the other letters had a similar pictorial origin. This means a marked advance on the Chinese system in invention and in economy in use, insomuch as a fewer number of characters, entering more freely into combination, suffice for a complete written language.

Symbols for numbers mark another advance; I, II, III seem obvious enough for the ideas "one," "two," "three," but it would clearly be very inconvenient to represent large numbers by a mere extension of the system; and so in early times the initial of a word was used as a symbol, thus: Gk. π, for πεντε represented "five," C for *centum*, represented a "hundred." But pictures were also used: thus five and ten were represented by V, a picture of the open hand, and X a double V. In Greek the letters of the alphabet, taken in order, were used for "one," "two," and so on, and the same letters with a ʹ for "ten," "twenty," and so on. There is, however, another aspect of the use of I, II, III worthy of the mathematician's notice; the picture form for "three men" would at first have been the picture of a man repeated three times, similarly, for three deer, etc., and the use of III with, but more especially without, a symbol for "man" or "deer" dissociates the number from the object and implies a perception of abstract number; while the use of the letters of the alphabet for numerals springs from an association of ideas between two sets of ordinals.

The Arabic numerals, to the European at any rate, are arbitrary and conventional symbols. Their origin, if it could be traced, would probably be found to be of a similar nature to the origin of those characters whose history we know; it is certainly none of the fanciful ones sometimes suggested to connect their form with their meaning. They are derived from Indian sources over a thousand years old, and were introduced into Europe by the Moors.

When once the symbols, arbitrary and conventional though they may seem, are adopted, we should expect that to form a symbol for "forty-two," it would be expedient to combine in some way those for "four," "ten," and "two." In point of fact the "ten" does not appear in the Arabic notation; a combination of 4 and 2 suffices. The combination might have been 24, 4^2, $\frac{2}{4}$, but it could not well have been 36; 42 may have been chosen to agree with

the arrangement of the abacus ; otherwise it is the result of an arbitrary choice among a limited number of possible forms. At any rate, the principle underlying it, once accepted, enables us to write down any number as a combination of the digits 1, 2, 3, . . . 9, 0, and conversely to read as a number any similar arrangement of the digits.

It is to be noted that the "ten" although apparently suppressed is there by implication, so that the form is shorthand for a somewhat complex arrangement ; but the resulting simplicity of form produces an economy in computation that has rendered the more obvious Roman form XXXXII obsolete for arithmetical purposes. It is also worth noting, as a corollary, that it is only in the decimal or denary scale that 42 stands for "forty-two." In a duodecimal system 36 would represent "forty-two," just as 3/6, meaning 3 shillings + 6 pence, stands for forty-two pence.

Other relations of "four" and "two" require other combinations of 4 and 2, and so the forms $2 + 4$, $2/4$, 4^2, $\sqrt[4]{2}$, have been invented. They express (as every diagram should express) vividly and quickly to the eye a definite notion ; having been found convenient, they have entered into use and survive.

To express similar notions in which the numerical values are general and not restricted, other symbols forming a new code were needed. But it took two thousand years to evolve the one we use.

As we have already remarked, the Greeks used a geometrical mode of representation for general magnitudes. But powerful as this method proved, in their development of it, for some purposes, it was inadequate for others. The possession of even so good a method may have militated against the invention of a better. We see now that no invention was needed ; the alphabet provides an extraordinarily convenient set of symbols, possessing two conspicuous advantages over a specially invented cipher : (1) the forms of the letters are familiar to the eye, (2) their names are familiar to the ear.

The examples of algebraical symbolism given at the end of the chapter will show that the introduction of the system we use came not from a spontaneous recognition of these advantages but by a process of evolution extending over many centuries.

Now it is obvious that the adoption of a code works in two ways : it serves for a means of communication among the initiated ; it is a source of mystification for the uninitiated. The tradesman's cipher conveys information to his assistant and preserves his secret from the customer ; "worth" is intelligible to an Englishman, but baffling in pronunciation and meaning to a Frenchman ; "q" is recognizable by Western Europeans, but is meaningless to Greeks and Russians ; a child may understand the ordinary use of letters and numerals sufficiently well to give a meaning to "by 2 axes," and be misled on seeing "$by + 2ax$," or may even know

all that is meant and implied by "3A Green St., S.E. 29," and only be mystified by $(p+q)^2$; a student of logic knows the code on which is based "if all A is B and all B is C, then all A is C," without necessarily having the vaguest idea of the significance of $CuSO_4.5H_2O$. And the disadvantage of symbols is this: that a mind capable of following the most intricate reasoning or versed in the most abstruse developments of a subject might be shut out from the ideas of his fellow-scientists through ignorance of the symbols they use. A mathematician might be able to follow the rhetorical algebra of the mediaeval schoolmen and fail before a simple equation in a modern textbook.

There must be countervailing advantages; there are. Firstly there is economy in writing and reading; $\frac{3}{4}$, $\odot$, $\angle$ ABC are quicker for hand to write and eye to see than "three-fourths," and "circle," "the angle between two intersecting lines (to be further specified)." With the development of a subject or even in an elementary case of some intricacy the rhetorical form of expression may become so involved as to be intolerable; while an attempt to express $\sqrt{-1}$ (so important in modern mathematics) in ordinary language will make it clear to what point we have come in economy of representation.

There is clearly, too, a further economy in mental attention, and even, as experience shows us, in processes of thought. Initiation into and a general agreement in the meanings of symbols and their combinations are necessary, just as communication by written or spoken English depends on an agreement and education in the sound and forms, and of course considerable practice is necessary too. We become so well acquainted with the appearance of ordinary handwriting that we take in the meaning of a paragraph without really reading it, whereas we should be compelled to spell out slowly and laboriously the same paragraph in back-handed writing. We perform, owing to long practice, the simple operations of arithmetic almost automatically; to perform the same operations on the same numbers in another system of notation—such as the duodecimal—would present the same sort of difficulties as reading back-handed writing.

But when initiation and practice have rendered us thoroughly familiar with a set of symbols, they make communication not only possible but extraordinarily simple. And in Mathematics the use of symbols brings an unexpected reward. The development of the subject itself proceeds from the manipulation (according to fundamental laws) of the symbols, introducing new ideas for consideration and interpretation, as we shall presently see.

In generalizing arithmetic, the form that indicated an operation or relation has usually been preserved in Algebra, thus $\frac{a}{b}$ and $x-y$ are general forms corresponding to the particular $\frac{2}{7}$ or $8-3$. (There is one conspicuous exception: ab is not the general form

of 42.) But algebra passes beyond arithmetic and the form $a - b$ leads to the idea of negative number and magnitude; the use of a^{π} demands an interpretation, if one can be found to satisfy the laws of algebra, of $x^{\frac{1}{2}}$ and y^{-3}.

Again

$$x = b - y, \text{ derived from } y + x = b,$$

and $$x = y^{-1} b, \quad ,, \quad ,, \quad yx = b,$$

suggest the analogous notation,

$$x = \sin^{-1} b, \text{ derived from } \sin x = b.$$

There are cases where convenience has adopted forms which to the beginner would be misleading; $f(x)$ does not mean $f \times x$, nor can d be cancelled in $\frac{dy}{dx}$.

There are symbols, again, which are only just being established or generally accepted, such as $n\,!$, which is more convenient to print than $\underline{|n}$; $\bar{3}\cdot4771$; $\overline{AB} + \overline{BC} = \overline{AC}$ and $\rightarrow \infty$ to express ideas which are entering more and more into mathematical work or theory. And as the subject develops in new directions, new symbols are being demanded; thus even in elementary arithmetic a need is felt for shorthand to represent the phrases "nearly equal to," "greater (or less) than but nearly equal to." Such symbols as ⑊ and ⫽ are suggestive enough and convenient enough to serve.

In the historical notes which follow, dates and names are freely given, largely to show the period in which Algebraical Symbolism settled down, the diversity that preceded a general acceptance of the symbols, the wide area of contribution from which they were drawn, and the authority of the form finally accepted. It will be seen that absolute agreement has not yet been reached in every particular.

The so-called Arabic numerals are of Indian origin found in inscriptions of the tenth century A.D. in a form somewhat like the Gobar Arabic numerals (about A.D. 1100 ?) from which our modern forms are presumed to be derived. Their convenience is largely due to the invention of a symbol for "zero," which is known to occur in a Gwalior inscription of A.D. 876.

The Arabic system was explained in an arithmetic of RABBI BEN EZRA (*b.* Toledo, 1097; *d.* Rome, 1167), and was brought into general use by the Liber Abaci of LEONARDO FIBONACCI (son of Bonacci, of Pisa; *b.* about 1175), published 1202.

The forms of the numerals found in MSS. and printed books varied somewhat until about A.D. 1500; since then they have remained practically unchanged; and, though there are still slight differences, as, for example, between the English and French script forms for 5 and 7, the characters used are almost universal.

In Geometry the Greeks used letters to specify the angular

points of a figure, the two end-point letters for a line segment and so on.

Descartes introduced the use of algebraical equations for geometrical loci A.D. 1637.

The graphic representation of complex forms such as $3 + 2\sqrt{-1}$ is due to Argand (1806), and is named after him, but the principle had been anticipated by Caspar Wessel in 1797.

In arithmetic the decimal system is almost certainly due to counting on the fingers of both hands. This was preceded by counting in 5's, one hand only being used. The latter is still common among primitive peoples; it is discernible in the Greek πεμπάζω (*pempazo*) and πεμπάζομαι, as in Homer's "Odyssey" IV, 411–12, *φώκας μέν τοι πρῶτον ἀριθμήσει καὶ ἔπεισιν αὐταρ ἐπὴν πᾶσας πεμπάσσεται* . . . "First he would count the seals and go over them again and when he had counted them all . . ."

The invention and retention of the symbol V is a piece of evidence of counting in "fives"; and the modern Welsh numerals, as well as the Cambrian sheep-counting numbers, retain traces of it; thus the Welsh numerals 16 to 19 are expressed as $1 + 5 + 10$ (*een a pump ddeg*), etc.

There are other relics of counting in "twenties" (derived from the use of both hands and both feet) in "fourscore," "quatre-vingts."

In the Roman numerals addition is indicated by juxtaposition as in VI, XXXII; and subtraction by juxtaposition with an inversion of the order as in IV, IX, a method suggesting an anticipation of the modern idea of directed number. In such arithmetical forms as $2\frac{1}{2}$, simple addition (of 2 and $\frac{1}{2}$) is indicated, while in 32 something more than simple addition is implied. But in algebra, as we have mentioned above, juxtaposition, following Descartes' notation, is used for multiplication.

Signs of Operation, etc.: $+$, $-$ were first used regularly by Johannes Widmann in his Mercantile Arithmetic (Leipzig, 1489). They were used at first to mean "excess" or "defect," and are most plausibly explained as being derived from warehouse marks on cases of goods, used to denote an excess over or defect from their intended weight.

Hoecke in a work published at Antwerp in 1514 and Stifel in his Arithmetica Integra (1544) used them, the latter occasionally as symbols of operation; but Vieta (François Viète, 1540–1603) was the first to use them consistently with their modern meaning.

$\times$ was first used by Oughtred (in his Clavis Mathematicæ, 1631) and Harriott (1560–1621).

The full stop . to indicate multiplication was introduced by Harriott. The Hindus had used it for subtraction.

$\div$ was used by Johann Heinrich Rahn at Zurich (1659) and John Pell in London (1668); according to a conjecture of

Rouse Ball, it is a combination of — (used by the Arabs for division) and :

The forms $\frac{\mathbf{a}}{\mathbf{b}}$, **a**/**b**, as well as **a** — **b**, were used for division by the Arabs.

= was chosen for a symbol of equality by ROBERT RECORDE because "noe 2 things can be more equal." It appeared in his WHETSTONE OF WITTE (1557). Vieta and others used = for the "difference between," as we use $\sim$. For a long time mathematicians, including NEWTON (1642–1727), used $\propto$ or ∞, which are regarded as transformations of the æ of æqualis.

∞ for "infinity" was introduced by WALLIS in his ARITHMETICA INFINITORUM (1655).

> and < are due to HARRIOTT and have held the field against OUGHTRED'S $\sqsupset$ $\sqsubset$

$\neq$, $\ngtr$, $\nless$ were used by EULER (1707–1783); they are little used now except by English mathematicians.

( ) was used by ALBERT GIRARD in his INVENTION NOUVELLE EN ALGÈBRE. STIFEL used ( )( ), and VIETA used the **vinculum.**

$\sqrt{}$, probably a form of **r** (for *radix*), was used by NICOLAS CHUQUET in LE TRIPARTY EN LA SCIENCE DES NOMBRES (1484). He also used $\sqrt[3]{}$, etc.

For proportion OUGHTRED in his CLAVIS MATHEMATICÆ used **a . b : : c . d**, instead of the **a** — **b** — **c** — **d** which was current in his time; but in his TRIGONOMETRY (1657) he introduced the form **a : b :: c : d**.

:: was freely used by BARROW (1686).

EULER is responsible for **e**, **i**, π, though OUGHTRED had used π, and to EULER is also due the notation **a**, **b**, **c**, **A**, **B**, **C**, in trigonometrical formulæ for the elements of a triangle.

The **decimal point** is due to JOHN NAPIER of MERCHISTON (1550–1617), the inventor of logarithms; and according to Rouse Ball the introduction of our decimal notation is due to BRIGGS (1561–1631), a great admirer of Napier and the first to compile common logarithms (i.e. to base 10). A comparison with the notations of STEVINUS (Simon Stevin, 1548–1620) 4 ⓪ 6 ① 2 ② 8 ③, or 46′ 2″ 8‴, shows the economy of the form 4·628.

, for marking off thousands, millions, etc., was used by LEONARDO OF PISA. The French now use it for the decimal point, and mark off thousands, etc., by a wider spacing of the digits.

Algebraical Symbols.—The use of letters of the alphabet for symbols occurs once in LEONARDO'S LIBER ABACI; but VIETA'S IN ARTEM ANALYTICAM ISAGOGE (*Introduction to Analysis*), 1591, is the first work in Symbolic Algebra. In it he used **B**, **C**, **D** and other consonants for known, vowels for unknown, magnitudes; **Aq**, **Ac**, **Aqq** (q for *quadratus*, c for *cubus*) where

we should use x^2, x^3, x^4. The forms current in his day were **R** (for *radix* or *res*) or **N** (for *numerus*) as equivalent of **x**; **z** (*zensus*), **C** (*census*) or **Q** (*quadratus*) as equivalent to $\mathbf{x}^2$; and **C** or **K** (*cubus*) as equivalent to $\mathbf{x}^3$.

Stifel used **1A**, **1AA**, **1AAA**; Harriott used **a**, **aa**, **aaa**, while Herigonus (of Paris) in his Cursus Mathematicus (1634–1637) wrote **a**, **a2**, **a3**.

Descartes (1596–1650) the inventor of Co-ordinate Geometry used $\mathbf{x}''$, $\mathbf{x}'''$ and $\mathbf{x}^2$, $\mathbf{x}^3$. He also introduced the use of **a**, **b**, **c**, for known, and **x**, **y**, **z** for unknown, magnitudes.

The use of a general index $\mathbf{a}^n$ is due to Newton (1642–1727).

Nicholas Oresmus (*b.* Caen, 1323; *d.* Lisieux, 1382), in his Algorismus Proportionum, had introduced the idea of fractional indices, and Stevin had also suggested their use.

Wallis was the first to interpret fractional and negative indices, using methods of interpolation to do so.

The abbreviations **sin**, **tan**, **sec** are due to Girard (1626), and **cos** and **cot** to Oughtred, but they were forgotten till Euler revived them and brought them into general use.

The following examples of notations will give a sort of bird's-eye view of the development of symbolic algebra, which began with syncopated forms, as *cu* for *cubus*, and proceeded to modern Symbolism. They are borrowed from the works of various authorities (such as Fink and Rouse Ball):—

Diophantus (Alexandria, third century A.D.).

$$\kappa^{\hat{\upsilon}}\ \bar{\alpha}\ \overline{ss}\ \bar{\eta}\ ⋔\ \delta^{\hat{o}}\ \bar{\epsilon}\ \mu^{\hat{o}}\ \hat{\alpha}\ \iota\ s\ \bar{\alpha}$$

for $$x^3 + 8x - (5x^2 + 1) = x.$$

This is syncopated algebra, the symbols being abbreviations of words:

$\kappa_{\hat{\upsilon}}$ for cubus; $\bar{\alpha}$ (the first letter of the alphabet) for 1; juxtaposition for addition; $\overline{ss}$ (plural) for x; $\bar{\eta}$ for 8 (the 8th letter in the alphabet); ⋔ (from λείψει) for —; $\delta^{\hat{o}}$ (δυναμις = power) for square; $\bar{\epsilon}$ for 5; $\mu^{\hat{o}}$ (μονας) for a unit; ι (ισος) for =; ς for x, $\bar{\alpha}$ for 1.

Leonardo of Pisa (1202) used res for x; censo, or ce, for x^2; cubo, or cu, for x^3; censo de censo, or cece for x^4; primo relato for x^5; censo de cubo, or cecu, for x^6; secondo relato for x^7.

Regiomontanus (Johann Müller of Königsberg, 1436–1476) used 16 census et 2000 æqu 680 rebus,

for $$16x^2 + 2000 = 680x$$

Chuquet (1488) used 12^1, 12^2 for $12x$, $12x^2$.

Cardan (1545) used cubus *p̄* 6 rebus æqualis 20,

for $$x^3 + 6x = 20.$$

Bombelli (1572) used 1 $\underset{\smile}{2}$ *p*. 5 $\underset{\smile}{1}$ *m*4 for $x^2 + 5x - 4$.

Stevin (1586) used 3 ③ + 5 ② − 4 ① + 6 for $3x^3 + 5x^2 - 4x + 6$.

Vieta (1591) used 1C − 8Q + 16N æqu 40

for $$x^3 - 8x^2 + 16x = 40;$$

and a cubus + b in a quadr 3 + a in b quadr 3 + b cubus æqualia
a + b cubus

for $$a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3.$$

REYMERS used

$$\begin{array}{ccccccccccccc} \text{XXVIII} & \text{XII} & & \text{X} & & \text{VI} & & \text{III} & & \text{I} & & \text{O} \\ \text{1 gr.} & 65532 & + & 18 & \div & 30 & + & 18 & + & 12 & + & 8 \end{array}$$

for $$x^{28} = 65532x^{12} + 18x^{10} - 30x^6 + 18x^3 + 12x + 8$$

BÜRGI (1552–1632) used

$$\begin{array}{ccccccc} \text{II} & & \text{IV} & & \text{VI} & & \text{VIII} \\ 16 & - & 20 & + & 8 & - & 1 \end{array}$$

for $$16x^2 - 20x^4 + 8x^6 - x^8.$$

DESCARTES (1637) used $zz \propto az - bb$ for $z^2 = az - b^2$.

CHAPTER VIII

NOMENCLATURE

Be not careless in deeds, nor confused in words, nor rambling in thought.—MARCUS AURELIUS.

THE terms used in Mathematics may be divided according to their derivation into three groups: (1) words of English origin, (2) words of Latin origin, (3) words of Greek origin.

The words of English origin are usually names of weights and measures, e.g. yard, furlong; they are to be regarded rather as words in common use than mathematical terms.

The words of Latin origin were nearly all introduced in the Middle Ages (say from 1100–1700), but especially subsequent to the Renaissance (say from 1500–1700), when Latin was the *lingua franca* of the intellectual world. Newton and his contemporaries wrote their works in Latin, and the new words they introduced for new ideas and new methods were usually of Latin origin, e.g. differential and integral calculus.

The words of Greek origin are usually geometrical terms, e.g. diagonal, diameter, and are a legacy of Euclid's geometry.

To study the meaning of words like quadrilateral, intercept, transversal, in the light of their origin and in relation to other words of kindred derivation, will not only make for clear thinking in mathematics but will also form a continued lesson in language.

Here will be given some notes on words especially interesting.

Weights and Measures.—The human body provided a number of units of measurement: the foot, the hand, the ell, the span, the cubit, the pace.

The **mile** (Lat. *mille passus*) was 1,000 double paces of about 30 inches each, and therefore was about 5,000 feet long.

The word **yard** is Anglo-Saxon for a stick. It is found in Chaucer with this meaning, e.g.:

> " or if men smoot it with a yerde smerte "
> (i.e. " if men struck it sharply with a stick.")
> (Prologue 148).

Robin Hood's arrows were a cloth-yard long, i.e. as long as the stick used for measuring cloth. Till quite recently drapers used a yard-stick for measuring lengths; it was probably originally the distance from the mouth to the tip of the fingers when the arm

was extended sidewise. I have seen drapers measuring calico by using this distance from mouth to finger tip without any other measure. (Cf. the yard or yard-arm of a ship; cf. also Fr. *verge* and *vergue.*) The **rod, pole,** and **perch** are all sticks. (Cf. Fr. *saut à la perche* = pole-jump.) **Rood** is a doublet of rod, and is to be found in "Holy Rood" and "By the Rood."

Furlong is furrow long, i.e. the length of a furrow.

Acre is an Anglo-Saxon word cognate with "ager," and meaning a piece of tilled land; originally as much as a yoke of oxen would plough in a day, this was standardized as an area of 32 furrows, each a furlong in length, which requires a furrow of breadth a little more than two feet.

An actual **chain** with its 100 **links** is still used by surveyors. An acre is 10 square chain-measures.

The word **pound** is short for *libra pondo*, i.e. a pound by weight.

Ounce is from the Latin *uncia*, $\frac{1}{12}$ of a lb., and **inch** is derived from the same word.

Grain is from *granum*, a small seed. Fr. *grain* and *graine.*

A **scruple** is a little pebble.

Stone was originally an actual stone.

Dram and **drachm** are from Gk. *drachma.*

Dwt. is **d** for penny and **wt.** for weight, just as **cwt.** is **c** for 100 and **wt.** for weight.

Ton or **tun** (A.-S.) was a large butt.

Gallon means a large bowl (cf. balloon, a large ball; galleon, a large galley).

Bushel is derived from a Low-Latin word meaning a little box.

Quart and **quarter** need no explanation.

The derivation of **pint** is uncertain; it may have been a "painted" bowl.

Avoirdupois means "possessions of weight," from *avoir*, "to have," and so "possessions"; *du* and *pois* (= *poids*).

Troy is from Troyes, the name of a town.

The fundamental weights and measures must have been variable; as all men have not feet of the same size, sticks are of different lengths, and stones of different weights. As long as only one draper, and only one miller served a neighbourhood, and they used always the same stick for measuring their wares and the same stone for weighing their flour, this did not matter; but as trade widened, it became necessary to standardize the weights and measures.

To-day the standard yard is the length at 62° F. between marks on two gold plugs in a brass bar in the walls of the Houses of Parliament; all other yards are copies of this. All pound-weights are copies of a standard pound. The gallon is a vessel which contains 10 lb. of pure water at 62° F.

These are settled by law and all other measures are copies, certified and from time to time examined officially.

There are thus separate units for length, weight and capacity.

In the metric system (instituted April 7, 1795) the units of weight and capacity are derived from the unit of length, the metre. **A gramme** is the weight of a c.c. of water of maximum density, i.e. at 3·93° C. A **litre** contains 1,000 c.c. The metre itself was intended to be one ten-millionth part of the distance of the North Pole from the equator; it is actually rather less. Officially it is the length of a platino-iridium rod made by the celebrated physicist Borda.

The French International metric standards are kept at the International Bureau of Weights and Measures, in the Pavillon de Breteuil, Parc St-Cloud, at Sèvres.

It is worth remarking that the Romans kept standards of their weights and measures in the temple of JUNO MONETA (from which we get our words "money" and "mint"); but they did not succeed in preventing copies from being of different sizes.

£. s. d. are the initials of *libri, soldi, denarii* introduced in the Italian forms by the Lombards who settled in Lombard Street as money-lenders and bankers. Their sign was three golden disks, representing gold coin; it has survived in the three golden balls of the pawnbroker; but is now disappearing, following in the wake of the barber's pole, the chemist's mortar and pestle, and the tobacconist's snuff-taking Highlander.

Libri means pounds. *See* "pound" above (which is both a sum of money and a weight), and cf. Fr. *livre*, Ital. *lire*. It is connected with *libro*, to balance, as in libration, equilibrium, and *libra*, scales.

Soldi means solid coin. Cf. Fr. *sou*, *solde* (pay), the Welsh *swllt* (shilling), and the English "soldier," one who draws pay. From the same root is derived solder (i.e. to make solid).

Farthing is fourthing, a fourth part.

Zero and cipher are both derived from the arabic *sifr*. Cf. Fr. *chiffre*.

Score is from an A.-S. word, meaning to cut. The butcher still "scores" the piece of pork. Shire, shear, share, ploughshare, sheer (cut-away), short, etc., are all from the same root. Reckonings were kept on a tally-stick (from *tailler* = to cut; the same root as tailor). This was a stick split lengthwise into two, of which the seller kept one part and the buyer the other. When a record of payment had to be made the two pieces were fitted together and a notch was cut. When the final reckoning was made the two parts had to agree or tally. It gave the name tallyman to the pedlar. The system is still used by milkmen in country districts in France. *N.B.*—We still "score" at cricket, and only a few years ago runs were frequently called "notches."

Gross is great (i.e. a great dozen or a dozen of dozens).

Arithmetic is from Gk. *arithmos*, a number.

Geometry is from Gk. *ge*, the earth (as in geography, geology), and *metreo*, I measure (cf. diameter, gas-meter).

Trigonometry, from Gk. *tri*, three, *gonia*, an angle or corner, and *metreo*, i.e. triangle measurement.

Gonia is probably connected with Gk. *gonu* and Lat. *genu*, a knee, a thing which bends and forms an angle; it occurs also in diagonal and polygon. In the same way angle and ankle are cognate, each meaning a "little bend."

Algebra is from *algebr w'al mukabala*, meaning "restitution and simplification," the title of a work by Mohammed ibn Musa Alkwarizmi (about A.D. 830). In Spain *algebrista* is still used to mean a bone-setter (i.e. restorer).

Mechanics is Gk. *mechanice* (sc. *techne*), the science of machines.

Dynamics is from Gk. *dunamis*, force. Cf. dynamo.

Kinetics and Kinematics are from Gk. *kineo*, I move. Cf. kinematograph, cinema (in America "movies").

Statics is from Lat. *sto*, I stand. Cf. stationary, stable, etc.

Calculation is from Lat. *calculus*, diminutive of *calx*, a little pebble; the same root as calcareous, and Fr. *chaux* (lime). The word dates from the time before abstract arithmetic, when counting was done on the fingers or with pebbles.

Straight (Lat. *strictus*, drawn tight); the same root as strict, strait-jacket, stretch, straits, stringent, and Fr. *étroit*. Till recently a straight line was more commonly called a right line. Cf. Fr. *droite* and Ger. *Gerade*, but the Germans also use *Strecke*, i.e. a stretch.

Right angle and **rectangle** are from Lat. *rectus*, and have the same origin as upright, erect, rectitude, direct, and Fr. *droit* and *droite*.

Perpendicular (Lat. *pendeo*, I hang), a hanging plumb-line; for the idea cf. German *Senk-recht* (straight-sinking). The Germans also use *Lot*, cognate with lead (which in Latin is *plumbum*).

Vertical (Lat. *verto*, I turn). The vertex is the turning-point where, after ascending an upward slope, one begins the descent of a downward one; hence it means top. The same root is found in reverse, converse, inverse, transversal, etc.

Parallel (Gk. *para allelois*), meaning "alongside one another."

Circle, a little "circus" or round thing; the same word as Latin *circa* and *circum*.

Centre, is from Gk. *kentron*, a point, sting, ox goad.

Radius (Lat.), the spoke of a wheel; ray has the same origin. The Fr. *rayon* means all three—a ray (of light), a spoke, and a radius.

Arc and **chord,** a bow (Lat. *arcus*) and its string (Lat. *corda*).

Minute. A minute or small part (1) of a degree, (2) of an hour; from Latin *primae minutae partes.*

Second, from *secundae minutae partes,* a second small part, i.e. a small part obtained by a second division.

In **diameter** and **diagonal** *dia* means through. Diameter is a measurement through; the Germans call it *Durch-messer.* Diagonal is a line "through the angular" points (*see* Trigonometry).

Polygon is a "many angle-d" figure.

Sector and **Segment** are both derived from Lat. *seco,* I cut.

Hypotenuse, from Gk. *hupoteinouse,* the subtending line. *N.B.*—*sub* in Latin corresponds to *hupo* in Greek, as supposition is the word corresponding to hypothesis; *tendo,* corresponding to Gk. *teino,* means I stretch.

Isosceles, from Gk. *isos,* equal, and *skelos,* a leg. We have iso- for equal in isobar, isochromatic, etc.

Triangle is the figure formed by joining three points in pairs

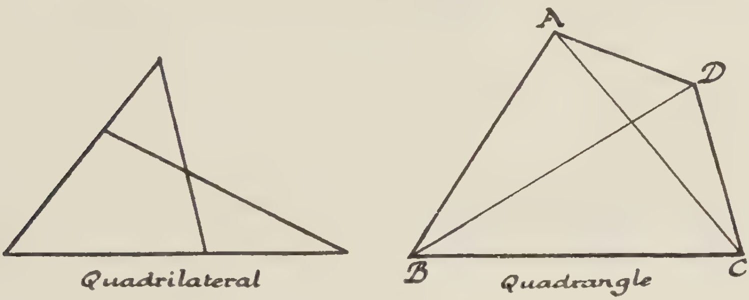


FIG. 97

by straight lines. **Quadrilateral** is the figure enclosed by four lines; a **quadrangle** in the mathematical sense is a figure formed by joining pairs of 4 points.

Point, from Lat. *pungo,* to prick, etc. From the same root we have pungent, poignant, puncture, punctuate and the French *point, pointe,* etc.

Fraction means a "breaking" (from Lat. *frango*). Fragile, refraction are from the same verb.

Factor means a "maker" (from Lat. *facio*); cf. factory, manufacture. Factors are numbers which when multiplied "make" or "produce" a given number. The number "produced" is the **product** of the factors.

Multiplication means "manifolding." The root of *plico* (I fold) gives us complex (wrapped up, involved), implicit, explicit (unfolded), three-ply, etc.; cf. Fr. *pli,* a fold and an envelope.

Subtract means "drawn away from" (from Lat. *sub* and *traho*). *Traho* is cognate with draw, *tractum* with draught; cf. detract, contraction, etc.

Denominator means the "namer" or classifier. In fractions it tells us what class of fractions we are dealing with. We speak of the denomination of coins, and of religious denominations. We cannot add coins of different denominations, thus: 3 florins and 4 crowns are not 7 of anything. To perform the addition we must reduce to coins of a common denomination, say shillings. The same rule holds for fractions.

Numerator is the "numberer," and tells us how many parts of a certain denomination there are. Number is derived from the Latin *numerus*, the intrusive *b* having been introduced for euphony; as in chamber, from *camera*; humble, from *humilis*. In the same way *d* is intrusive in tender, from *tener*; cinder, from *cinis, cineris*; *voudrai*, future of *vouloir*, etc.

Quadratic is from Latin *quadratus*, squared. From the same root we have squadron and quarry (a place where stone is squared), and the French have *équerre, carré*. The Germans use *quadrat* both in algebra and geometry for "square."

Eliminate (from Latin *e* and *limen* = a threshold) means to kick out of doors. From the same root we have limit and lintel.

Mean, from Lat. *medianum* (itself from *medius*), meaning middle, through the Fr. *moyen*, as in *Le moyen âge* = the Middle Ages. (For the loss of the *d*, cf. ray and Fr. *rayon*, from *radius*.)

Parabola, a section of a cone made by a plane parallel to a generating line OB. The word means a "putting side by side";

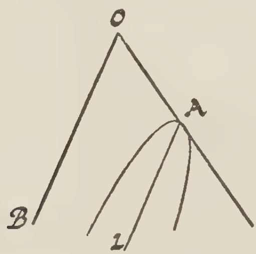


FIG. 98

and so a "parable" is a comparison. From parabola is derived *parabolare*, which becomes the Fr. *parler*, and gives us Parliament, parlour, and Dogberry's contemptuous "Palabras! neighbour Verges"; this survives in the modern form "palaver."

Ellipse is from Gk. *elleipsis*, a defect or falling short; is a section that has "failed to reach" the position of parallel. "Ellipsis" in grammar implies that something is missing.

Hyperbola is from Gk. *huperbole*, meaning excess; is a section that has gone beyond the position of parallel. Cf. hyperbole = excess in language. (See Fig. 99.)

Focus, a Latin word meaning hearth, whence rays of warmth and light radiate. From focus are derived the Fr. *foyer*, which means both "hearth" and "focus," and *feu* (cf. *jeu* from *jocus*, *lieu* from *locus*, *peu* from *paucus*). In the case of a double convex lens the use of the word is almost a pun. The Germans use *Brennpunkt*, i.e. burning-point, for focus.

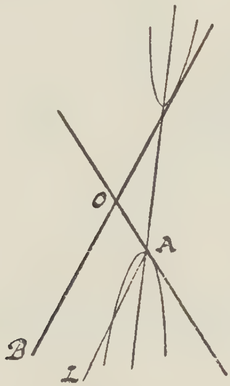


FIG. 99

Sine. The Arabic word bore a close resemblance to the word for "bosom," and by confusion was translated into the Latin *sinus*. Sinus means various curved things, e.g. a bay; and it is a little anomalous that it should be applied to a straight line, which is preferred to an arc as the measure to determine an angle. On the Continent "sinus" is used.

CHAPTER IX

SYMMETRY

THE human body possesses a property called symmetry. On each side of a central plane there is an eye, a leg, an arm. If E_1, L_1, A_1 are special points on the eye, leg, and arm on one side, E_2, L_2, A_2 corresponding points on the other, then for perfect symmetry the $\triangle$s $E_1L_1A_1$ and $E_2L_2A_2$ are congruent, and E_1E_2, L_1L_2, A_1A_2 are bisected at right angles by the central plane S—the plane of symmetry. Single features, the central line of the nose, the central line of the lips, lie in the plane of symmetry. This sort of symmetry is called bilateral symmetry.

The symmetrical properties of the isosceles triangle and circle were among the first geometrical truths to be discovered.

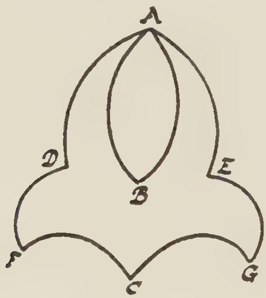


FIG. 100

We can say of a plane figure which is bilaterally symmetrical that all single points (such as A, B, C in the figure), lie on one straight line, which is the axis of symmetry ; all pairs of corresponding points (D and E, F and G, etc.) can be joined by straight lines DE and FG, etc., which will be bisected at right angles by the axis of symmetry. This property is used by draughtsmen. Thus, if the left-hand side of the figure is drawn, perpendiculars to the axis can be drawn from chosen points and produced till the whole line is bisected by the axis, and the end points thus obtained are on the right-hand half of the figure. The right-hand side is then drawn in with the help of these points.

Or if one half of the figure is drawn, folding along the axis of symmetry will permit tracing to be used for the other half.

Again, consider a symmetrical figure, such as the isosceles triangle drawn on a card white in front, black behind. If the isosceles triangle is cut out and turned round back to front it will fill the aperture. In geometrical language, if a symmetrical figure

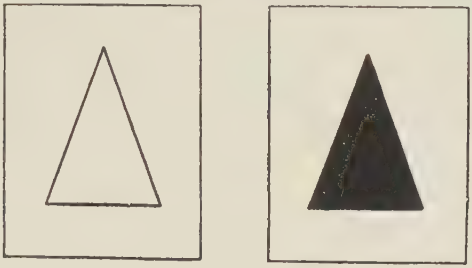


FIG. 101

be rotated about its axis of symmetry though half a complete revolution its new position will coincide with the old one. In this rotation it sweeps out a solid of revolution. Thus, an isosceles triangle generates a right circular cone; a rectangle, a right circular cylinder; a circle, a sphere. All solids turned truly on a lathe are solids of revolution.

There are plane figures not bilaterally symmetrical which appeal to the eye as having a shapeliness that we might call symmetrical. The parallelogram is one. It is right to apply the word

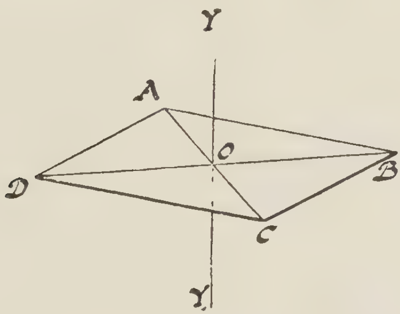


FIG. 102

symmetrical (which means co-measurable) to such cases for

AB = opposite side CD,
AD = opposite side BC;
∠B = opposite angle D,
∠A = opposite angle C.

Now, taking a line YOY_1 through O, the intersection of the

diagonals, and perpendicular to the plane ABCD, consider ABCD to be rotated in its own plane about YOY_1 as axis. After half a complete revolution ABCD will coincide with its original position. ABCD then has a certain kind of symmetry which we may call **central symmetry.**

Defining symmetry in terms of revolution about an axis, we can use almost the same wording for bilateral and central symmetry. In the first case, however, the rotation is outside the plane of the figure about an axis in its plane; in the second it is in the plane about an axis perpendicular to the plane. In each case a straight line joining a pair of symmetrical points is bisected at right angles by the axis of symmetry.

Some figures have both sorts of symmetry, e.g. the square, the rectangular hyperbola. There are also figures, such as the regular polygons and some curves, which possess both bilateral symmetry and what we might call multiple central symmetry.

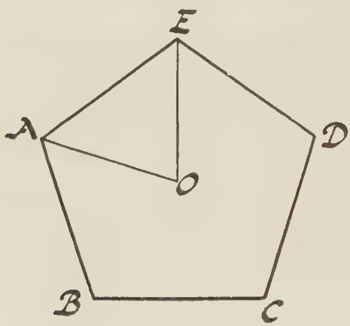


FIG. 103

Consider a regular pentagon ABCDE, with O the centre of the circumcircle. The figure possesses bilateral symmetry about a number of axes—AO, BO, etc. If it is rotated about a right axis through O, then after a fifth of a complete revolution its new position coincides with ABCDE, A being on B, B on C, etc., E on A.

Before entering upon the considerations to which these ideas lead us we may remark that if these properties were established as fundamental propositions, a number of others would follow as immediate deductions, e.g. establishing the property of the kite ABCD (Fig. 104) by regarding it as made up of two isosceles $\triangle$s ABD, BCD, we should have the figures for the bisection of lines and the drawing of perpendiculars to lines as particular cases.

We may also note, in passing, that a regular n-gon has n axes of bilateral symmetry. For after each cyclic rotation through $\frac{360°}{n}$ in its own plane it coincides with its original position, and

thus any one axis of symmetry is an axis for n different positions, or n different lines are in turn axes of symmetry.

Every bisector of an angle, and every right bisector of a side, is also an axis of symmetry. But that does not give $2n$ axes, for if n is even, the right bisectors of sides coincide in pairs and the

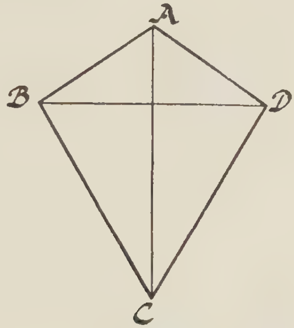


FIG. 104

bisectors of angles coincide in pairs ; while if n is odd, each right bisector of a side coincides with the bisector of an angle.

The circle may be regarded as a polygon in which n is infinitely great, i.e. there are an infinite number of axes of symmetry, and if the figure is rotated, it coincides with its original position for any angle of revolution—a property of great practical importance.

In graphs or Cartesian geometry a figure is bilaterally symmetrical about the axis OX if the indices of y in its equation, are even, and about the axis OY if the indices of x are even. For take the simple case $y = x^2$.

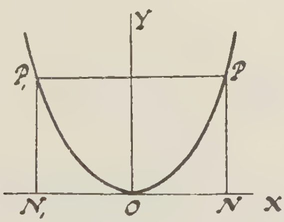


FIG. 105

If P and P_1 are symmetrical points and P is (x_1, y_1) then

$$P_1 \text{ is } (-x_1, y_1) ;$$

for $$P_1N_1 = PN \text{ and } ON_1 = -ON,$$

and $$y_1 = x_1^2 \text{ and also } = (-x_1)^2,$$

and the substitution of $-x$ for x in the equation does not alter the value of y.

The circle $x^2 + y^2 = a^2$ and the hyperbola $x^2 - y^2 = a^2$ are symmetrical about both axes (Fig 106).

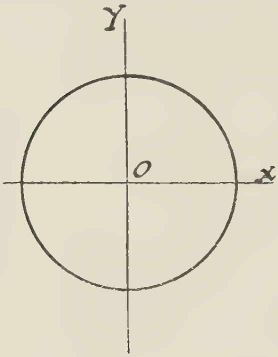


FIG. 106 (1)

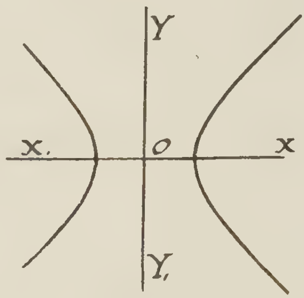


FIG. 106 (2)

For central symmetry, the terms of the equation are all of even, or all of odd, degree in x and y.

Thus, $y = x^2$ is not cyclically symmetrical; $x^2 + y^2 = a^2$ and $x^2 - y^2 = a^2$ are.

If P and P_1 are two points cyclically symmetrical with respect to a right axis through O, then $OP = OP_1$, and P and P_1 are in opposite quadrants, i.e. if P is $(x_1, y_1,)$ P_1 is $(-x_1, -y_1)$.

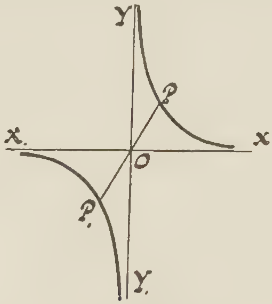


FIG. 107

If $-x$ and $-y$ are substituted for x and y the equation will not be altered if all terms are of even degree; but it will if some are of odd degree. Let the equation of the rectangular hyperbola passing through P and P_1 be $xy = c^2$.

Then, if P satisfies it,

$$x_1y_1 = c^2;$$

i.e. $$(-x_1)(-y_1) = c^2;$$

i.e. P_1 satisfies it.

To obtain a line bilaterally symmetrical, about OY, with a given line, substitute $-x$ for x in its equation.

Thus $y = mx + c$ and $y = -mx + c$ are bilaterally symmetrical with each other about OY, they intersect on OY and with XOX they form an isosceles triangle.

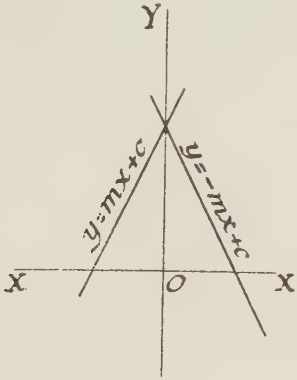


FIG. 108

To obtain a line centrally symmetrical with a given line with respect to a right axis through O, substitute $-x$ and $-y$ for x and y.

Thus $y = mx + c$ and $-y = -mx + c$ are centrally symmetrical with each other. They are parallel and cut OX and OY in pairs of points equidistant from O, i.e. the four points in which they cut the axes are the angular points of a parallelogram.

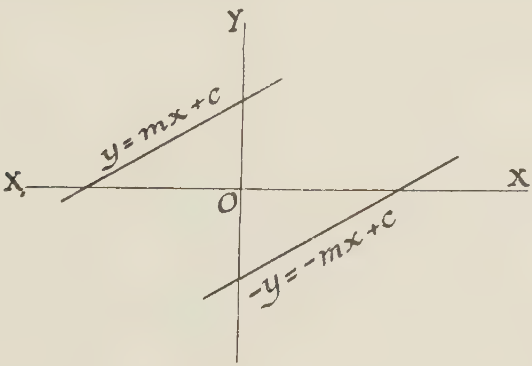


FIG. 109

Symmetry in Algebra.—To both types of symmetry algebra furnishes analogies :

$$\downarrow$$

Thus $ax^4 + bx^3 + cx^2 + bx + a$ may be regarded as a function

$$\uparrow$$

whose coefficients possess bilateral symmetry, c, the single feature, being on the axis, the coefficients of x^3 and x being symmetrically placed, i.e. equidistant from the axis, and so for the coefficients of x^4 and x^0. A rotation about the axis will reverse the function without altering the order of the coefficients $a + bx + cx^2 + bx^3 + ax^4$.

Powers of functions possessing this property have the same property; thus the binomial $a \overset{\downarrow}{\underset{\uparrow}{+}} b$ is a simple bilaterally symmetrical function, the axis being between the terms.

The powers

$$(a + b)^2 = a^2 + 2ab + b^2,\ (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

all exhibit the property, including the general positive integral expansion:

$$(a + b)^n = a^n + c_1a^{n-1}b + c_2a^{n-2}b^2 + \ldots + c_2a^{n-2}b^2 + c_1a^{n-1}b + b^n.$$

This statement, intuitively accepted, might be used to determine the coefficients of a number of powers: thus $(a + b)^3$ cannot have any other form than $a^3 + pa^2b + pab^2 + b^3$, where p is a number to be determined (*see* also the chapter on "Degree," p. 168).

Put $a = b = 1$;

then $\qquad 2^3 = 1 + p + p + 1$;

i.e. $\qquad p = 3$.

Similarly,

$$(a + b)^5 = a^5 + pa^4b + qa^3b^2 + qa^2b^3 + pab^4 + b^5.$$

Put $a = b = 1$;

$$2^5 = 1 + p + q + q + p + 1,$$

i.e. $\qquad 15 = p + q \quad \ldots\ldots\ldots\ldots \quad$ (I)

Put $a = 2$, $b = 1$.

$$3^5 = 32 + 16p + 8q + 4q + 2p + 1,$$

$$210 = 18p + 12q,$$

$$35 = 3p + 2q \quad \ldots\ldots\ldots\ldots \quad \text{(II)}$$

whence $\qquad p = 5, q = 10$.

For higher powers the number of substitutions required is the number of undetermined coefficients that are involved; for $(a + b)^n$ it is the integral part of $\frac{n}{2}$.

For even powers a negative substitution is convenient:

$$(a + b)^4 = a^4 + pa^3b + qa^2b^2 + pab^3 + b^4.$$

Put $a = b = 1$,

$$16 = 1 + p + q + p + 1$$

i.e. $\qquad 2p + q = 14 \quad \ldots\ldots\ldots\ldots \quad$ (I)

Put $a = 1$, $b = -1$,

$$0 = 1 - p + q - p + 1$$

$$q + 2 = 2p, \quad \ldots\ldots\ldots\ldots \quad \text{(II)}$$

whence $\qquad q = 6, p = 4$.

Again, if it is required to factorize a function bilaterally

symmetrical, the factors must either be in themselves bilaterally symmetrical or appear in pairs of which one is bilaterally symmetrical with the other.

Thus $$a^2 + 2ab + b^2 = (a + b)^2$$
$$2a^2 + 5ab + 2b^2 = (2a + b)(a + 2b),$$
the coefficients of $2a + b$ being the coefficients of $a + 2b$ in the reversed order.

If $3x^4 + px^3 + qx^2 + px + 3$ can be factorized, it can only be in one of the following ways:
$$(x + 1)(3x^3 + lx^2 + lx + 3),$$
$$(3x^2 + mx + 1)(x^2 + mx + 3),$$
$$(x^2 + nx + 1)(3x^2 + kx + 3).$$
Thus, $3x^4 + 5x^3 + 4x^2 + 5x + 3 = (x + 1)(3x^3 + 2x^2 + 2x + 3)$
$= (x^2 + 2x + 1)(3x^2 - x + 3)$;
$3x^4 + 8x^3 + 14x^2 + 8x + 3 = (3x^2 + 2x + 1)(x^2 + 2x + 3)$.

Factors of such expressions can be found by making the assumptions in all possible forms, multiplying out and substituting suitable values for x and y. (It may be convenient to substitute before multiplying out.)

Thus, take $3x^4 + 8x^3 + 14x^2 + 8x + 3$.

Try $(x + 1)(3x^3 + lx^2 + lx + 3)$.

Put $x = 1$;

then $$36 = 2(6 + 2l),$$
$$9 = 3 + l,$$
$$l = 6.$$
This is impossible, since if $l = 6$, 3 would be a factor of $3x^3 + lx^2 + lx + 3$ and therefore of the original expression.

Try $(x^2 + nx + 1)(3x^2 + kx + 3)$.

Put $x = 1$;

then $$36 = (2 + n)(6 + k) \quad \text{. . . . . (I)}$$
Put $x = -1$;

then $$4 = (2 - n)(6 - k) \quad \text{. . . . . (II)}$$
Simplifying (I) and (II) and subtracting,
$$32 = 4k + 12n.$$
$$8 = k + 3n.$$
Substituting in (II) for k
$$4 = (2 - n)(-2 + 3n),$$
i.e. $$3n^2 - 8n + 8 = 0,$$
of which there are no integral solutions.

Finally, try $(3x^2 + mx + 1)(x^2 + mx + 3)$.

Put $x = 1$;

then $$36 = (4 + m)^2,$$
$$\pm 6 = 4 + m,$$
$$m = 2 \text{ or } -10,$$
and -10 is impossible. ; i.e. the factors are
$$(3x^2 + 2x + 1)(x^2 + 2x + 3).$$

In most textbooks of algebra the solution of equations of the

type $3x^4 + 8x^3 + 14x^2 + 8x + 3 = 0$ is made to depend on its bilateral symmetry. The symmetrical terms are grouped together: $3(x^4 + 1) + 8(x^3 + x) + 14x^2 = 0$, and it is reduced to the form $3(x^2 + 1)^2 + 8(x^2 + 1)x + 8x^2 = 0$, which can then be solved:

$a^2b + b^2c + c^2a$ is an algebraical function which displays the property of cyclic symmetry in a, b, and c.

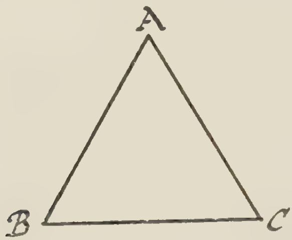


FIG. 110

An equilateral $\triangle$ ABC after a rotation through 120° about a right axis through its centre occupies the position BCA, after a second rotation the position CAB and after a third rotation it again occupies the original position ABC.

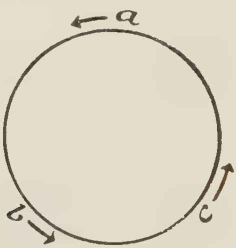


FIG. 111

If the symbols a, b, c be placed on the circumference of a circle at points of an inscribed equilateral $\triangle$, if and we consider the figure to rotate through angles of 120°,

for each rotation a moves on to b
,, ,, b ,, ,, c
,, ,, c ,, ,, a

These changes rotate $a^2b + b^2c + c^2a$
successively into $b^2c + c^2a + a^2b$ and $c^2a + a^2b + b^2c$,
and back to $a^2b + b^2c + c^2a$;
the whole function remaining unchanged may be called a cyclic group of terms, or a cyclic function.

A single term may possess the property if its factors exhibit

the property among themselves; thus abc, $(b-c)(c-a)(a-b)$ are unchanged by cyclic rotation.

Occasionally a function may be cyclically symmetrical with respect to two sets of symbols; thus $ax+by+cz$; $x^2(b-c)+y^2(c-a)+z^2(a-b)$ are cyclic functions of a, b, c, and of x, y, z.

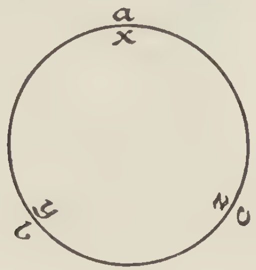


FIG. 112

In these examples it will be seen that a and x, b and y, c and z are associated. In $ax+by+cz$, x appears with a. In $x^2(b-c)+y^2(c-a)+z^2(a-b)$, x appears in the term which does not involve a; this expression could be written

$$x^2y^2z^2\left\{\frac{b-c}{y^2z^2}+\frac{c-a}{z^2x^2}+\frac{a-b}{x^2y^2}\right\},$$

thus bringing a and x, b and y, c and z into positional association.

A function possessing cyclic symmetry must have for factors, either (1) factors which are themselves cyclically symmetrical, or (2) factors which form cyclic groups.

Thus $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$ can be shown to have $a-b$ as a factor; it then follows that $b-c$ and $c-a$, the cyclic mutations of $a-b$, must be factors; as the function is of the third degree, there can be no other factor but a numerical one; let this be m.

Then

$$a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=m(b-c)(c-a)(a-b).$$

Let $a=0$.

Then $$bc^2-cb^2=m(b-c)(c)(-b),$$

but as bc^2-cb^2 factorizes as $-bc(b-c)$ our factorization is confirmed and m is seen to be 1.

Σ (sigma, Gk. capital S) and Π (pi, Gk. capital P) are used in sums and products for an abbreviated notation.

$a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$ is written $\Sigma a(b^2-c^2)$,

$(b-c)(c-a)(a-b)$ is written $\Pi(b-c)$,

one term or factor being written instead of three. This notation, in connexion with some simple results, such as $\Sigma(b-c)=0$, $\Sigma a(b^2-c^2)=-\Sigma bc(b-c)=-\Sigma a^2(b-c)=\Pi(b-c)$, effects a considerable economy in the manipulation of cyclic expressions.

A cyclic function of a, b, c whose degree is not a multiple of 3 must have a factor which is itself cyclically symmetrical.

Thus $\Sigma a(b-c)^3$ can be shown to have $a-b$ and therefore $b-c$ and $c-a$ as factors. But the expression is of the 4th degree, therefore there will be a numerical factor (which may be 1) and a factor of the 1st degree in a, b, and c. This must be $a+b+c$, the only 1st degree cyclic function (*see* also p. 168).

$$\therefore\quad \Sigma a(b-c)^3 = m\,(b-c)\,(c-a)\,(a-b)\,(a+b+c).$$

Let $a = 0$.

Then $$bc^3 - cb^3 = m\,(b-c)\,(c)\,(-b)\,(b+c),$$

i.e. $$bc(c^2-b^2) = m(-bc)\,(b^2-c^2).$$

This result confirms the factorization and gives $m = 1$.

$a^2+b^2+c^2+bc+ca+ab$ can have no factors; for being of the 2nd degree, it could only have $(a+b+c)$ as a factor; any other would require the two cyclic mutations as co-factors and thus produce a 3rd degree function; i.e. the only expression of the form $\Sigma a^2 + p\Sigma bc$ which factorizes is $a^2+b^2+c^2+2bc+2ca+2ab$, which is $(a+b+c)^2$.

In triangles whose elements are a, b, c; A, B, C; we have two sets of symbols, in which a of the first is associated with A of the second, b with B, and c with C. If a formula involves any of these symbols either it must be cyclically symmetrical itself or the cyclic mutations of the formula must also be true.

Thus, since $c^2 = a^2+b^2-2ab\cos C$;

by cyclic rotation we have,

$$a^2 = b^2+c^2-2bc\cos A,$$
$$b^2 = c^2+a^2-2ca\cos B.$$

Again, in $$\Delta = \sqrt{s(s-a)\,(s-b)\,(s-c)}\,;$$

Δ is independent of any particular side and should therefore be a function possessing cyclic symmetry. Now $s = \frac{1}{2}(a+b+c)$ is cyclically symmetrical, and so is the group $(s-a)\,(s-b)\,(s-c)$.

Again, $$2R = \frac{a}{\sin A}.$$

Now R is independent of any particular side or angle, and should therefore be a symmetrical function of a, b, c and of A, B, C, and would remain unchanged for cyclic mutations;

therefore $2R$ = each of the cyclic mutations of $\dfrac{a}{\sin A}$,

i.e. $$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The formula $R = \dfrac{abc}{4\Delta}$ shows R as a symmetrical function of a, b, and c.

If in a group of three equations each is the cyclic mutation of the others, the same property holds for the solutions.

Thus when it is found that $x = \dfrac{d}{a+b+c}$ in the solution of

$$ax + by + cz = d,$$
$$bx + cy + az = d,$$
$$cx + ay + bz = d,$$

it may be concluded that $y = z = \dfrac{d}{a+b+c}$.

When $x = b - c$ is found to satisfy

$$by + cz = a(c - b),$$
$$cz + ax = b(a - c),$$
$$ax + by = c(b - a),$$

it may be concluded that $y = c - a, z = a - b$.

The argument by symmetry is used in cases like the following:

If x, y, z are unequal and if $\dfrac{xyz}{y+z} - x^2 = \dfrac{xyz}{z+x} - y^2$, show that each $= \dfrac{xyz}{x+y} - z^2$.

Now
$$\frac{xyz}{y+z} - x^2 = \frac{xyz}{z+x} - y^2;$$
i.e.
$$xyz\left\{\frac{1}{y+z} - \frac{1}{z+x}\right\} = x^2 - y^2,$$
$$xyz\,\frac{x-y}{(y+z)(z+x)} = (x-y)(x+y);$$
and since $x \neq y$,
$$\frac{xyz}{(y+z)(z+x)} = (x+y),$$
i.e.
$$xyz = (y+z)(z+x)(x+y). \quad . \quad (\text{I})$$

The symmetrical character of this result shows that since $\dfrac{xyz}{y+z} - x^2 =$ one of its cyclic mutations, it also $=$ the other.

To confirm this take $\dfrac{xyz}{y+z} - x^2$;
substituting for xyz from (I) we have that this
$$= (x+y)(z+x) - x^2$$
$$= yz + zx + xy.$$
Making cyclic rotations we get that
$$\frac{xyz}{z+x} - y^2 \text{ and } \frac{xyz}{x+y} - z^2, \text{ each} = \Sigma yz = \frac{xyz}{y+z} - x^2.$$

When a group of simultaneous equations exhibits any property of symmetry or interchangeability, it is useful to work through this property to arrive at the solution.

Thus to solve
$$y + z = a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$z + x = b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$x + y = c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

There is cyclic symmetry in x, y, and z of a *sum* of two unknowns; *adding* (1), (2), and (3) we get

$$2(x + y + z) = a + b + c;$$

i.e. $$x + y + z = s. \quad \text{(I)}$$

a symmetrical result.

Subtracting (1), (2), and (3) in turn from (I), we obtain the solutions $x = s - a$, etc.

Again in
$$yz = a^2 \quad (1)$$
$$zx = b^2 \quad (2)$$
$$xy = c^2 \quad (3)$$

there is cyclic symmetry in x, y, z of *products* of pairs; multiplying (1), (2) and (3) we get

$$x^2y^2z^2 = a^2b^2c^2$$
$$xyz = \pm\, abc \quad \text{(I)}$$

a symmetrical result.

Dividing (I) by (1), (2), and (3) in turn, we obtain the solutions $x = \pm \dfrac{bc}{a}$, etc.

In the same way in manipulating functions it is often advisable to preserve any symmetrical properties that are in evidence.

Coming back to the geometry of the triangle we shall see that, as in trigonometrical formulæ, the cyclic property of its elements can be made use of.

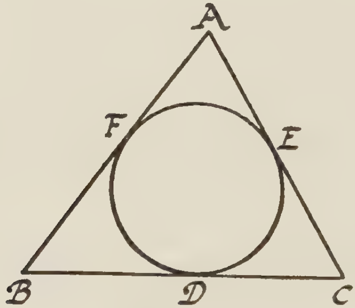


Fig. 113

If D, E, F are the points of contact of the inscribed circle it can be shown that AF = AE = $\frac{1}{2}(b + c - a)$, i.e. $b + c - a$ is a special function of magnitudes connected with a or A.

[For $b + c - a = (b + c + a) - 2a$ and $b + c + a$ is cyclically symmetrical.]

$$\therefore \quad BD = BF = \tfrac{1}{2}\,(c + a - b),$$
and $$CD = CE = \tfrac{1}{2}\,(a + b - c).$$

If H is the ortho-centre it can be shown that $AH^2 + a^2 = 4R^2$.

It follows that $BH^2 + b^2 = CH^2 + c^2 = 4R^2$, all these results being obtained by rotation of the groups A, B, C; D, E, F; a, b, c.

The functions $a + b + c$, $(b + c)(c + a)(a + b)$, and the formula $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ possess a property that $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$ and $(b - c)(c - a)(a - b)$ do not, viz. that they are unaltered for an interchange of a pair of

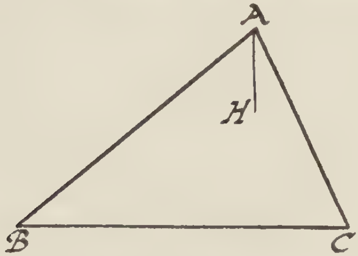


FIG. 114

symbols, e.g. a and b; we may say that these functions are symmetrical (not merely cyclically symmetrical) in a, b, and c.

This property of symmetry in functions extends to their factors taken either singly or in groups.

Thus, $a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + pabc$ is symmetrical in a, b, and c for all numerical values of p; it can factorize in one of two forms only:

$$(b + c)(c + a)(a + b) \quad \ldots\ldots\ldots\ldots \quad \text{(I)}$$
$$(a + b + c)(\Sigma bc) \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(II)}$$

These are obtained by considering that (1) if there are factors, one must be of the 1st degree, (2) there is no a^3, b^3, or c^3 in the expression, (3) they cannot contain a negative term, (4) no coefficient can be other than 1.

In (I) it follows that if $b + c$ is a factor, $c + a$ and $a + b$ must also be.

In (II) since $a + b + c$ is symmetrical, the other, the 2nd degree, factor must be of the form $bc + ca + ab$.

Putting $a = b = c = 1$,

we find that if p is 2, (I) gives the factors,

we find that if p is 3, (II) gives the factors.

If p is anything else there are no factors.

In equations if the property of interchangeability is displayed either in each equation independently or in a simultaneous group among one another, the property will also appear in the solution.

Thus one solution of

$$\left.\begin{aligned} x + y + z &= 6 \\ yz + zx + xy &= 11 \\ xyz &= 6 \end{aligned}\right\}$$

being found (possibly by inspection) to be $x = 1, y = 2, z = 3$, we can have the 6 permutations of this result, and the number

of solutions should be the product of the degrees, i.e. $1 \times 2 \times 3$. Therefore the table below gives the complete solution.

x	y	z
1	2	3
1	3	2
2	1	3
2	3	1
3	1	2
3	2	1

In
$$\left.\begin{aligned} ax^2 + bxy + cy^2 = d \\ ay^2 + bxy + cx^2 = d \end{aligned}\right\},$$
we get by subtraction $(a - c)(x^2 - y^2) = 0$;
i.e. $x = \pm y$, unless $a = c$; and solutions are
$$x = +y = \pm\sqrt{\frac{d}{a+b+c}};$$
$$\text{or} \quad x = -y = \pm\sqrt{\frac{d}{a-b+c}}.$$

In the formulæ
$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B,$$
there is interchangeability of A and B.

Lastly, to solve the pair of simultaneous equations $xy = 6$, $x + y = 7$, i.e. to find two numbers whose product is 6 and whose sum is 7. They are 6 and 1. The pair of equations can have only two solutions; they are
$$x = 6, y = 1 \text{ and } x = 1, y = 6.$$

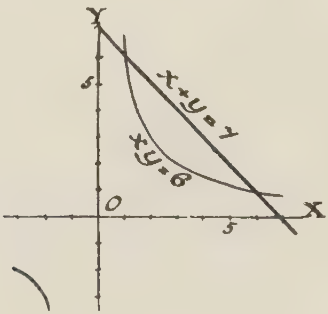


FIG. 115

If these are solved graphically we see that both graphs are symmetrical about the line bisecting $\angle$ XOY.

For let P_1 be a point (a, b), and let P_1N_1 and P_1M_1 be perpendiculars to the axes.

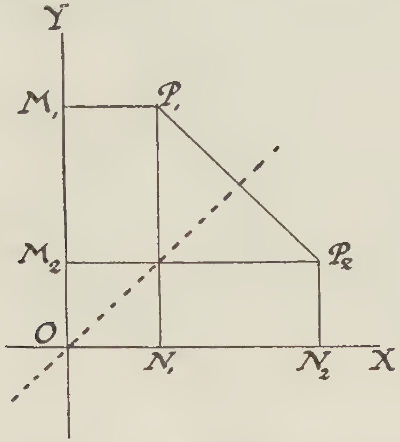


FIG. 116

Consider the figure to be rotated about the bisector of $\angle$ XOY, then
OY falls along OX and M_1 falls on N_2, where $ON_2 = OM_1 = b$;
OX falls along OY and N_1 falls on M_2, where $OM_2 = ON_1 = a$;
P_1 will fall on P_2, the intersection of perpendiculars at N_2 and M_2, and its co-ordinates are (b, a). That is, P_1 and P_2 are symmetrical about the line $y = x$ if the co-ordinates of P_1 and P_2 are interchangeable in x and y.

Some of the properties of the ortho-centre of a triangle provide a specially interesting application of the idea of interchangeability.

Let $A_1A_2A_3$ be a triangle, A_1P_1, A_2P_2, A_3P_3 the perpendiculars meeting in A_4.

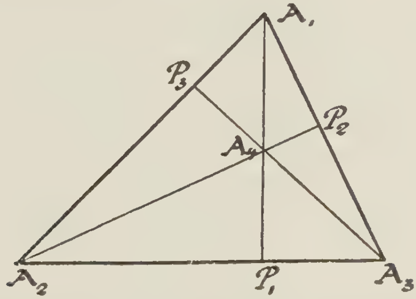


FIG. 117

A_1 can be shown to be the ortho-centre of $A_2A_3A_4$; similarly A_2 is the ortho-centre of $A_3A_4A_1$, and A_3 of $A_4A_1A_2$; P_1, P_2, P_3 being the pedal points for each triangle.

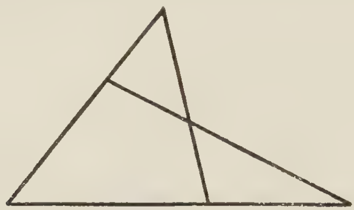
Any relation depending on the right angles at P_1, P_2, P_3 and connecting groups of A's will hold for all interchanges of suffixes. Thus, as we have already seen, (p. 141) $A_1A_4^2 + A_2A_3^2 = A_2A_4^2 + A_3A_1^2 = A_3A_4^2 + A_1A_2^2$, a relation in A's in which all the suffixes are involved interchangeably.

Again, the quadrilateral $P_1A_3P_2A_4$ is cyclic. We have here two P's and two A's alternately, and all the suffixes.

$P_2A_1P_3A_4$, and $P_3A_2P_1A_4$ are also cyclic.

The quadrilateral $A_3P_2P_3A_2$ or $P_3A_2A_3P_2$ is cyclic. Here again there are two A's and two P's, with only two suffixes which alternate; $A_1P_3P_1A_3$ and $A_2P_1P_2A_1$ are also cyclic. These six include all interchanges involving two P's and two A's.

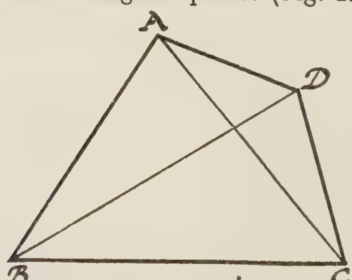
Again the circle $P_1P_2P_3$ can be shown to pass through the mid-point of A_2A_3; it will also pass through the mid-points of A_3A_1, A_1A_2, A_1A_4, A_2A_4, A_3A_4, these involving all interchanges of the suffixes. This circle is the 9-point circle.



Quadrilateral

Fig. 118

Two straight lines meet in one point, two points are joined by one straight line. The words straight line and point are interchangeable for properties of intersection. Thus since n straight lines may have $\frac{n(n-1)}{2}$ points of intersection, n points may give $\frac{n(n-1)}{2}$ straight joining-lines. A complete quadrilateral has six angular points (Fig. 118). A quadrangle ABCD (Fig. 119),



Quadrangle

Fig. 119 (1)

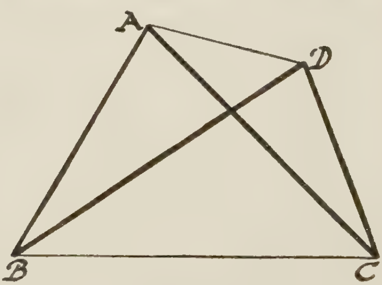


Fig. 119 (2)

has six sides (of which two are usually termed diagonals). Thus

BD and AC are diagonals of ABCD in Fig. 119 (1). . . . . (I)

but we may also say that

BC and DA are diagonals of the skew quadrilateral ABDC in Fig. 119 (2) . . . . (II)

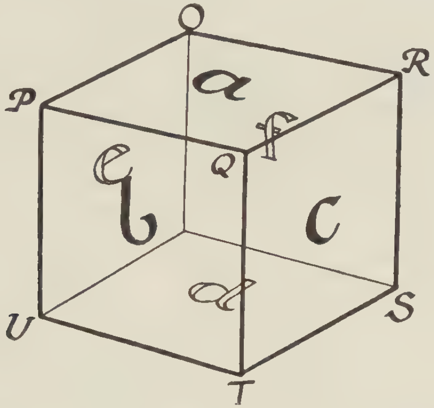


FIG. 120

and AB and CD are diagonals of the skew quadrilateral ACBD (III)

The interchange of C and D or of B and A in (I) gives (II) and the interchange of B and C or A and D in (I) gives (III).

For properties of intersection the words line and plane and point and plane are also interchangeable.

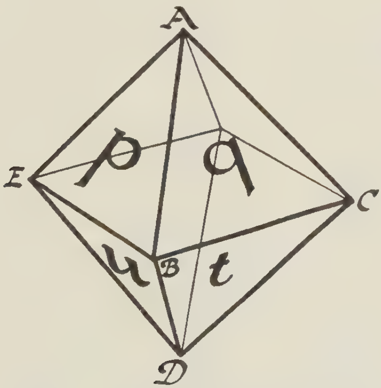


FIG. 121

Thus two straight lines determine a plane; two planes meet in a straight line. Three points determine a plane. Three planes meet in one point.

These correspondences can be seen in the regular solids in the

cube and the octahedron. With capital letters for apices and small letters for planes (heavy type for visible lines and planes, fine for invisible ones), we can use the notation (**ab**) for the line of intersection of the planes **a** and **b**, (**abc**) for the point where the planes **a**, **b**, **c** meet; just as we use **AB** to represent the line joining the points **A** and **B**, and **ABC** to represent the plane determined by the points **A**, **B**, and **C**.

Again, in the cube (**abc**) is **Q** and in the octahedron **ABC** is **q**. In the octahedron (**qtsr**) is **C**, in the cube **QTSR** is **c**. In the cube (**bc**) is **QT**, in the octahedron **BC** is (**qt**).

This inversion is also seen in the case of the dodecahedron, Fig. 122 and icosahedron, Fig. 123; with similar lettering and notation **ABCDE** is **z**, (**abcde**) is **Z**; (**xzu**) is **B**, **XZU** is **b**.

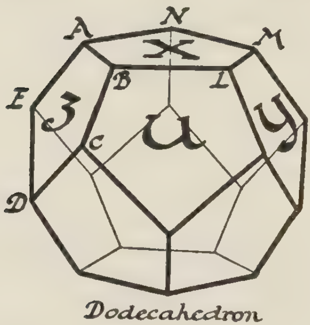


FIG. 122

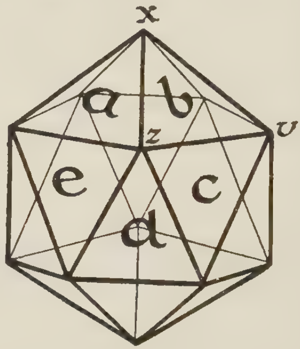


FIG. 123

The belt of faces **zuy** . . . corresponds to the belt of points **ZUY** . . . ; the zigzag belt of points **EDC** . . . corresponds to the belt of planes **edc** . . .

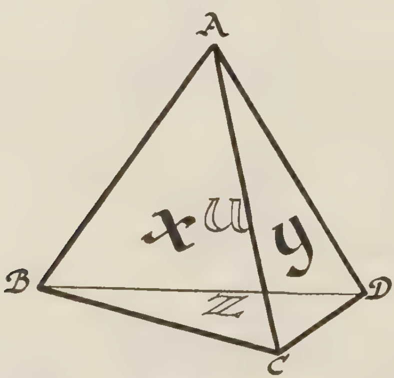


FIG. 124

The same inversion is seen in a pair of tetrahedra suitably placed.

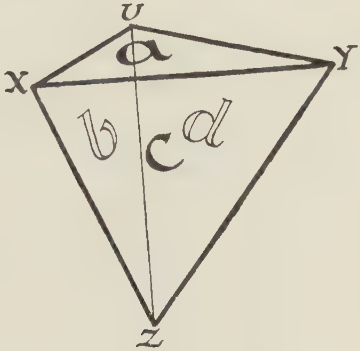


FIG. 125

In these examples chosen from the regular solids the most salient correspondence is that of point and plane. Connected with it are other remarkable examples of interchangeability.

First, tabulating the numbers of edges, faces, and angles, we have

—	Edges	Faces	Angles
Tetrahedron	6	4	4
Cube	12	6	8
Octahedron	12	8	6
Dodecahedron	30	12	20
Icosahedron	30	20	12

It will be observed that the cube and octahedron form a pair having the same number of edges, but having the numbers of faces and angles (i.e. planes and points) interchanged. The ratios of these numbers 3 : 4 or 4 : 3 connect (1) the numbers of faces meeting at an angle—3 for the cube, 4 for the octahedron ; (2) the numbers of edges to one face—4 for the cube, 3 for the octahedron. The dodecahedron and icosahedron form another pair exhibiting the same incidence of the numbers 3 and 5, which are (1) the numbers of faces meeting at an angle for the dodecahedron and the icosahedron, and (2) the numbers of edges of one face of the icosahedron and dodecahedron.

It will be interesting to see whether these ratios 3 : 4 for the first pair and 3 : 5 for the second are involved in the mensuration

of the solids, and whether relations of the one pair correspond to relations of the other pair, and whether, to bring the tetrahedron into line with these pairs, two tetrahedra may be regarded as a pair.

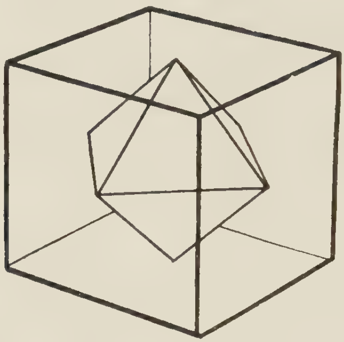


FIG. 126

The mid-points of the faces of a cube are the angular points of an octahedron (Fig. 126); and the mid-points of the octahedron are the angular points of a cube (Fig. 127). To a face of one corresponds an angle of the other. To show directly that an angle of one cor-

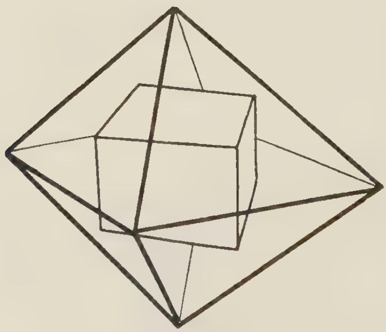


FIG. 127

responds to a face of the other, imagine the cube (or the octahedron) cut away at each angle by planes equally inclined to all the edges which meet at the angle; in this way the octahedron (or cube) can be formed.

These remarks apply equally to the dodecahedron and

icosahedron. They also apply to a pair of tetrahedra as in Fig. 128.

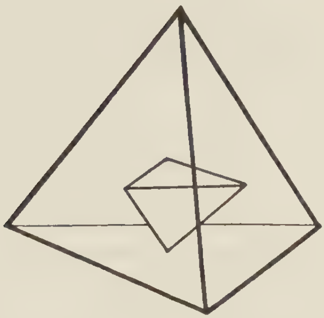


FIG. 128

A cube and an octahedron can be so combined (to form a solid of inter-penetration) that each edge of one and one edge of the other bisect at right angles (Fig. 129). This would not be possible if they had not the same number of edges.

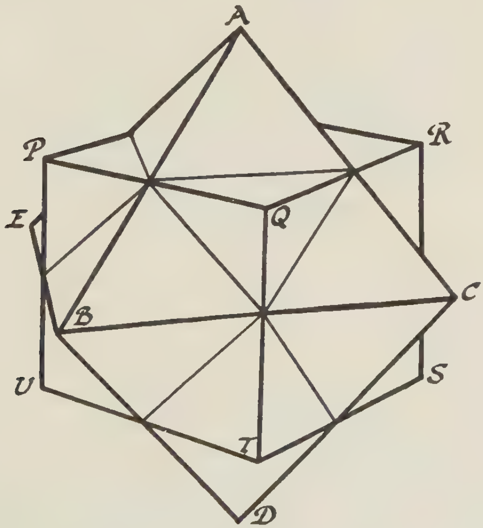


FIG. 129

As each apex of one corresponds to a face of the other, it is the apex of a pyramid whose base is in the corresponding face; there are 6 pyramids on square bases with A,B,C, . . . as apices,

and 8 on triangular bases with P,Q,R, . . . as apices. The common part of the two solids is shown in Fig. 130.

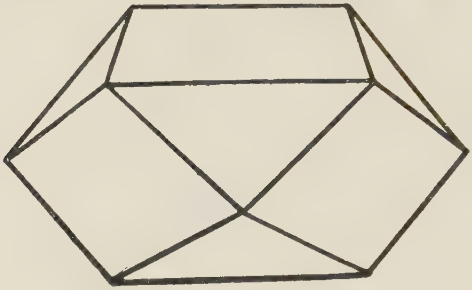


FIG. 130

There is a similarly formed solid of inter-penetration for the dodecahedron and icosahedron* (Fig. 131). The pyramids

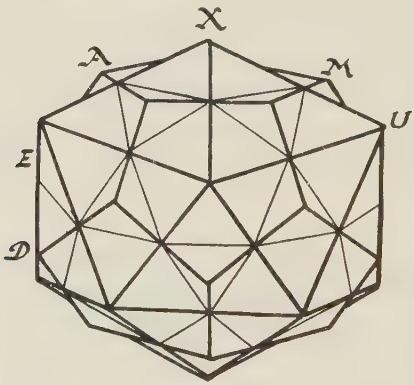


FIG. 131

projecting from the common solid are 20 on triangular, 12 on pentagonal bases.

Two equal tetrahedra will form a solid of inter-penetration.

The surfaces and volumes of cubes and octahedra (1) inscribed in the same circumsphere of radius R, (2) circumscribed to the same in-sphere of radius r, (3) having equal spheres that pass through the mid-points of edges and of radius ρ (this is the sphere circumscribing the common part of the solid of inter-penetration), can be easily calculated ; they are given by the following table :

* The skeleton of the common part of the two solids (i.e. the solid of Fig. 131 shorn of its pyramidal projections) is used decoratively in the War Memorial Exhibition at the Crystal Palace.

Surface		Volume	
Cube	Octahedron	Cube	Octahedron
$8R^2$	$4R^2\sqrt{3}$	$\frac{8}{3\sqrt{3}}R^3$	$\frac{4}{3}R^3$
$24r^2$	$12r^2\sqrt{3}$	$8r^3$	$4\sqrt{3}r^3$
$12\rho^2$	$8\rho^2\sqrt{3}$	$2\sqrt{2}\rho^3$	$\frac{8\sqrt{2}}{3}\rho^3$

Now if S_6, S_8, V_6, V_8 represent the surfaces and volumes of the cube and octahedron, and if α_4 and α_3 are written for $\frac{360°}{4}$ and $\frac{360°}{3}$, we see that

(1) for a common R, $S_6 : S_8 = V_6 : V_8$
$= \sin\alpha_4 : \sin\alpha_3$,

(2) for a common r, $S_6 : S_8 = V_6 : V_8$
$= \sin\alpha_4 : \sin\alpha_3$,

and (3) for a common ρ $S_6 : S_8 = \sin\alpha_3 : \sin\alpha_4$
and $V_6 : V_8 = \sin^2\alpha_3 : \sin^2\alpha_4$.

We also see that if the two solids have equal circumspheres they have equal inscribed spheres, and vice versa.

The calculations for the dodecahedron and icosahedron are not so simple, but they give the same relations for the surfaces and volumes in the different cases, when the suffixes are suitably changed.

These results are obtained immediately from the table :

Surface		Volume	
12-hedron	20-hedron	12-hedron	20-hedron
$\frac{10R^2}{\sin\alpha_5}$	$\frac{5R^2\sqrt{3}}{\sin^2\alpha_5}$	$\frac{20R^3\cos^2 36°}{3\sqrt{3}\sin^2\alpha_5}$	$\frac{10\,R^3\cos^2 36°}{3\sin^3\alpha_5}$
$\frac{15r^2\sin\alpha_5}{2\cos^4 36°}$	$\frac{15\sqrt{3}r^2}{4\cos^4 36°}$	$\frac{5r^3\sin\alpha_5}{2\cos^4 36°}$	$\frac{5\sqrt{3}r^3}{4\cos^4 36°}$
$\frac{15\rho^2}{2\cos^2 36°\sin\alpha_5}$	$\frac{5\rho^2\sqrt{3}}{\cos^2 36°}$	$\frac{5\rho^3}{2\cos 36°\sin^2\alpha_5}$	$\frac{10\rho^3}{3\cos 36°}$

There remain to consider cases of restricted interchangeability,

i.e. cases where the interchangeability is only applicable to some of the variables.

Where a, b, c, A, B, C are the elements of a $\triangle$ we saw that such functions as the area and the circumradius, since they did not depend on one side or angle in any way differently from the others, should be and were interchangeable functions of the elements. In the same way r (the in-centre) $= \dfrac{\Delta}{s}$.

But the ex-centres correspond severally to the angular points (or their opposite sides)—thus $r_1 = \dfrac{\Delta}{s-a}$; being especially associated with a or A, it is a special function of them; but b and c remain interchangeable.

Consider again any function of any special angle or side, such as $c^2 = a^2 + b^2 - 2ab\cos C$.

$a^2 + b^2 - 2ab\cos C$ is an expression for c^2, and therefore involves C specially, but a and b interchangeably.

$\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$, the expression for $\tan\dfrac{A}{2}$ should be a special function of a, and it is—in the numerator by exclusion and in the denominator by inclusion; b and c are interchangeable, as $\tan\dfrac{A}{2}$ depends on each in the same sort of way.

In $\sin A + \sin B + \sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$; both sides of the identity are symmetrical.

But in $\sin A + \sin B - \sin C = 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2}$; each side involves C specially, A and B interchangeably.

The reader's experience will confirm the importance and extend the usefulness of these ideas, which belong to the nature of mathematics.

Finally, we may note that wherever ideas of symmetry can be applied they will simplify the solution of a problem.

E.g. ordinarily it would be better to choose $x - 1$, x, $x + 1$ than to choose x, $x + 1$, $x + 2$ for three consecutive integers. In analytical geometry the simplest equation of the circle is obtained if two diameters are chosen as axes; if a problem is connected with two points A and B, it is probable that the most convenient axes are AB (produced) and the right bisector of AB; if a problem is connected with two straight lines AOB and COD, the most convenient axes are the bisectors of the angles at O.

In cases of loci a good deal of information may be obtained or a definite line of inquiry suggested by consideration of positions of symmetry.

Thus, " Given O a fixed point, XY a straight line, and a line OQ drawn to any point Q of XY, and divided at P so that OP . OQ is constant, say c^2 ; to find the locus of P."

Draw OA perpendicular to XY. This is an axis of symmetry for the data and therefore for the locus.

Now A is one position of Q. Let B be the corresponding position of P, i.e. OB . OA = c^2. Now since OA is a definite length, so is OB. Therefore B is a fixed point.

It can now be proved that PBAQ is cyclic since OP . OQ = OA . OB ; that BPQ is a right angle ; that OPB is a right angle ; and therefore that the locus of P is the circumference of a circle on OB as diameter.

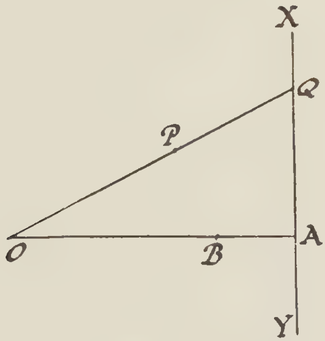


FIG. 132

See also p. 229 (Special Cases).

But there is a class of problems in which a position of symmetry is not merely a help ; it is the solution. In a great number of cases, at any rate, the position of a maximum or a minimum is a position of symmetry. A general discussion cannot be given, but a number of cases, which the reader can easily increase for himself, may be mentioned.

Thus for all triangles on the same base : (1) of those having the same area, the isosceles has the least perimeter ; (2) of those having the same vertical angle, the isosceles has the greatest area. Of all polygons of a given number of sides, and having equal perimeters, that which has the greatest area is the regular polygon. If in the plane of a given circle, centre O, a point P be taken, the greatest and least lines that can be drawn to the circumference lie along PO ; the greatest chord that, produced if necessary, passes through P is a diameter that is part of PO produced ; the shortest chord passing through P when P is within the circle is perpendicular to PO, i.e. of these four lines three lie along the axis of symmetry PO, and the fourth is symmetrical with respect to PO. In an ellipse the greatest and least diameters are the axes

of symmetry ; if S and H are the foci, the greatest angle subtended by SH is at the extremities of the short diameter (minor axis), the least angle (it is zero) is at the extremities of the long diameter (major axis).

Here is the sort of argument on which our general statement depends :—

Let a magnitude change according to the position of a variable point P.

Let XY be an axis of symmetry for all positions of P, and let P_1 and P_2 be two symmetrical positions of P very near to XY.

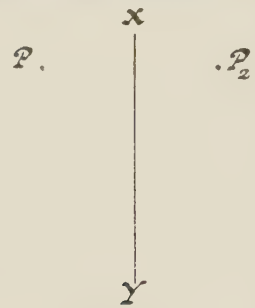


FIG. 133

Now if the magnitude increases or decreases as P approaches XY from P_1, by symmetry it also increases or decreases as P approaches XY from P_2, and there is probably a maximum or minimum value of the magnitude for a position of P on XY. If there were a maximum or minimum value of the magnitude for a position of P, say P_3, nearer to XY than P_1 there would be a position of P, say P_4, symmetrical with P_3, for which the magnitude would have an equal maximum or minimum value, and the argument applied to P_1 and P_2 would now apply to P_3 and P_4.

This does not prove the general statement, but it establishes a strong reason for looking for maxima and minima in symmetrical positions. The proof in each case must depend on the data.

CHAPTER X

ANALOGY

It is no longer considered necessary in elementary teaching to separate one branch of mathematics from another—algebra is admitted into arithmetic and geometry; graphs are used everywhere. But it is perhaps insufficiently recognized how in the history of mathematics the knowledge of one branch has helped the development of another and how an illustration from one branch may illuminate a piece of work in another.

It is not surprising that the Greeks should discuss algebraical theory such as the Solution of Equations and the Summation of Series by geometrical methods, or that Newton should have put his Dynamics into the form of geometrical propositions; but it is remarkable that Archimedes should have reversed the process and deduced geometrical results from considerations of equilibrium.

Euclid in dividing a line into two parts so that the square on one should equal the rectangle contained by the whole and the other part was virtually solving the equation $x^2 = a(a - x)$,

FIG. 134

where a is the whole line and x the first part; and he used a method which corresponds step by step with the algebraic method of solving a quadratic by completing the square.

Take the present textbook solution, which is in essentials the same as Euclid's. AB is a; BC, at right angles, is $\frac{1}{2}a$. AC is joined, and CD = BC. Then AD is x.

The equation is

$$x^2 + ax = a^2 = AB^2.$$

Add $(\frac{1}{2}a)^2$;

$$x^2 + ax + (\tfrac{1}{2}a)^2 = a^2 + (\tfrac{1}{2}a)^2 = AB^2 + BC^2$$
$$= AC^2$$

[In geometry two squares are added by Pythagoras' Theorem.]

$$\therefore \quad (x + \tfrac{1}{2}a) = AC$$

Take away $\frac{1}{2}a$;

$$x = AC - \tfrac{1}{2}a = AC - CD$$
$$= AD.$$

Euclid's method, however, gives only one solution; the algebraic gives two and suggests how the other may be geometrically obtained.

The complete solution of the more general equation

$$x^2 \pm ax = b^2$$

may be obtained from an adaptation of the above figure.

Let AB represent b; BE, at right angles to it, represent a.

On BE as diameter describe a circle, and let its centre be C. Join AC, meeting the circumference in D_1 and D_2.

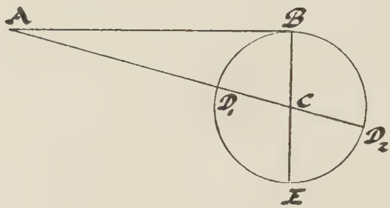


FIG. 135

Then the solutions of $x^2 + ax = b^2$ are AD_1, $-AD_2$; and of $x^2 - ax = b^2$ are $-AD_1$ and AD_2.

Now AB is a tangent to the circle;

$$\therefore \quad AB^2 = AD_1 \,.\, AD_2 = AD_1\,(AD_1 + D_1D_2),$$

i.e. $\qquad b^2 = x(x + a)$, where x is AD_1;

or $\qquad b^2 = AD_2 \,.\, AD_1 = AD_2(AD_2 - D_1D_2)$

$$= -x(-x - a)$$

$$= x(x + a), \text{ where } -AD_2 = x;$$

and similarly for the equation $x^2 - ax = b^2$.

Considering the theory of the roots of a quadratic, if x_1 and x_2 are the roots of $x^2 + ax - b^2 = 0$,

then $\qquad x_1 + x_2 = -a \quad \ldots\ldots\ldots\ldots \quad (1)$

and $\qquad x_1x_2 = -b^2 \quad \ldots\ldots\ldots\ldots \quad (2)$

AD_1 being x_1 and $-AD_2$ being x_2, we have

$$AD_1 - AD_2, \text{ which } = -D_1D_2 = -a,$$

$$AD_1 \,.\, AD_2 = b^2 = AB^2; \text{ and this is so.}$$

Algebraical methods are, as we see here and in the following pages, particularly suitable for an analysis of the possibility and the nature of the solution.

In the same way $x^2 \pm ax + b^2 = 0$ can be solved geometrically, thus (Fig. 136):

Let AB be a.

At B erect a perpendicular BC of length b. Through C draw CQ_1Q_2 parallel to AB meeting the circumference of the semicircle on AB in Q_1 and Q_2.

Through Q_1 draw Q_1P_1 perpendicular to AB.

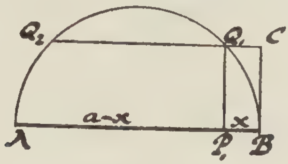


FIG. 136

Let BP_1 be x, then P_1A is $a - x$.

And since $$AP_1 . P_1B = Q_1P_1^2 = BC^2,$$
$$x(a - x) = b^2,$$
i.e. $$x^2 - ax + b^2 = 0.$$

It is left to the reader to consider the nature of the roots algebraically and geometrically, to consider the solutions obtained by using Q_2 or by letting AP_1 be x, and to see that the connexion between the solutions of $x^2 + ax = b^2$ and $x^2 - ax + b^2 = 0$ is contained in the discussion on pp. 205–6.

Consider again the problem: *To divide a given line of length* a *into two parts, such that the sum of the squares on the parts equals the square on another given line of length* c.

If x is one part, $a - x$ is the other;

and $$x^2 + (a - x)^2 = c^2,$$

i.e. $$x^2 - ax = \frac{c^2 - a^2}{2} \quad \text{. . . . . (I)}$$

This can be solved geometrically by the general method for solving
$$x^2 - ax = b^2, \text{ if } c > a,$$
or $$x^2 - ax + b^2 = 0, \text{ if } c < a.$$
There is, however, a special solution which will be given below.

Now, if the point of division is internal, then $x < a$, and $x^2 - ax$ is negative, $\therefore \frac{c^2 - a^2}{2}$ is negative, i.e. $c < a$.

Again completing the square in (I),
$$x^2 - ax + \frac{a^2}{4} = \frac{2c^2 - a^2}{4};$$
and for real solutions $2c^2 - a^2$ must be positive, i.e. $c > \frac{a}{\sqrt{2}}$.

That is, real solutions by internal division are possible only if $\frac{a}{\sqrt{2}} < c < a$.

[What happens in the special cases $c = \frac{a}{\sqrt{2}}$ or $c = a$?]

The reader may similarly discuss the values of c for which real external division is possible.

Again, if x_1 and x_2 are the solutions of (I), then, by the Theory

of Quadratics, $x_1 + x_2 = a$, i.e. the line is divided so that both solutions are obtained together.

The special geometrical solution is as follows:

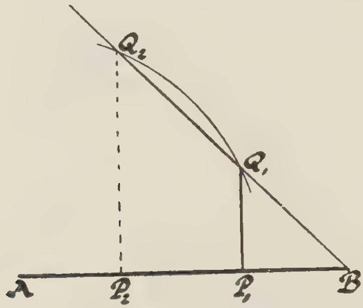


FIG. 137

Let AB be the given line of length a; at B draw BQ_1Q_2 so that $\angle ABQ_1 = 45°$.

With A as centre and a line of length c as radius describe an arc cutting $BQ_1 Q_2$ in Q_1 and Q_2. Draw Q_1P_1 perpendicular to AB. Then the line is divided as required at P_1.

It is left to the reader to show from this construction that the foot of the perpendicular from Q_2 gives another point of division but not different lengths of the parts, and that the results of the algebraical analysis are confirmed.

It must not be imagined, however, that the algebraical analysis, illuminating as it may be, always leads to the most direct or most elegant solution of a geometrical problem. Consider this:

Let A, B, C, D be fixed points in a straight line. It is required to find in ABCD a point O such that AO . OC = BO . OD.

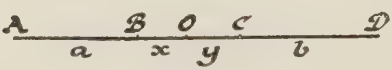


FIG. 138

Suppose O to be found.

Let AB be a, BO x, CO y, and CD b units of length.

Then
$$(a + x)y = x(y + b),$$
$$ay = bx,$$
$$\frac{x}{y} = \frac{a}{b},$$
i.e. BC is to be divided in the ratio AB : CD, and this can be done in many ways.

Following the algebraical hint, we should obtain the geometrical analysis

$$AO \,.\, OC = BO \,.\, OD,$$

$$\frac{AO}{BO} = \frac{OD}{OC},$$

$$\frac{AO - BO}{BO} = \frac{OD - OC}{OC},$$

$$\frac{AB}{BO} = \frac{CD}{OC},$$

i.e. $$\frac{BO}{OC} = \frac{AB}{CD}.$$

N.B.—It is clear that AD and BC are divided in the same ratio.

But the equality of the rectangles AO . OC and BO . OD suggests the following construction, which the algebraical analysis in no way anticipates :

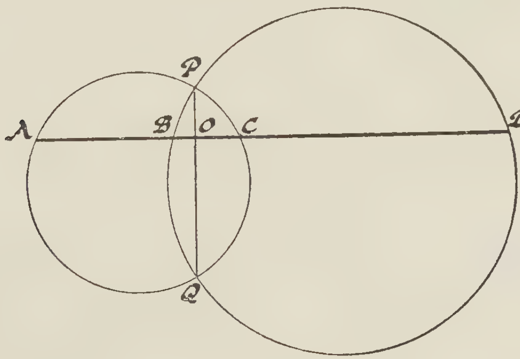


FIG. 139

Draw any two circles of which AC and BD are respectively chords. They will intersect in real points, say P and Q.

Let PQ meet AD in O. Then O is the point required.

For $\quad AO \,.\, OC = PO \,.\, OQ = BO \,.\, OD,$

a solution which is more direct.

In these problems, in which the algebraic and geometric solutions correspond, we have something more than analogy : we have the same problem solved by the same method but with different sets of symbols.

Some cases will now be given of the use of real analogy, e.g., in whatever way multiplication corresponds to addition,

division will correspond to subtraction,
evolution will correspond to multiplication,
involution will correspond to division.

These correspondences are seen in the relation of the manipulation of numbers to the manipulation of logarithms. They are also seen in the relation of geometrical to arithmetical progressions.

Thus in A. P. the law is one of addition,

in G. P. the law is one of multiplication.

Where subtraction, multiplication, division occur in A. P., we should look for division, raising to a power, finding a root in G. P. Thus u_n of an A. P. is $a + (n - 1)d$; then to obtain u_n of a G. P., for multiplication by $n - 1$ we should take an $(n - 1)$th power; for the + we should take × in G. P.; and so u_n of a G. P. should be of the form ad^{n-1}.

The arithmetic mean of x and y is $\frac{1}{2}(x + y)$. To the addition $x + y$ should correspond the product xy, and to the $\frac{1}{2}$ should correspond $\sqrt{}$ in G. P., and so the geometric mean should be $\sqrt{xy}$.

Or take the question, "To form a sequence, of which two terms and the law are known."

Let the 3rd and 6th terms of (1) an A. P., (2) a G. P., be m and n. To determine the progressions.

With the usual notation we have

A.P.	G.P.
$m = a + 2d \quad \ldots \ldots (1)$ $n = a + 5d \quad \ldots \ldots (2)$	$m = ar^2 \quad \ldots \ldots (1)$ $n = ar^5 \quad \ldots \ldots (2)$
By subtraction, $n - m = 3d$	By division $\dfrac{n}{m} = r^3$
By division by 3, $\dfrac{n - m}{3} = d$	By taking 3rd root, $\sqrt[3]{\dfrac{n}{m}} = r$
Multiplying by 2, $\dfrac{2}{3}(n - m) = 2d$	Taking 2nd power, $\sqrt[3]{\dfrac{n^2}{m^2}} = r^2$
Subtracting from (1): $a = m - \dfrac{2}{3}(n - m)$ $a = \dfrac{5m - 2n}{3}$	Dividing into (1): $a = m \div \sqrt[3]{\dfrac{n^2}{m^2}}$ $a = \sqrt[3]{\dfrac{m^5}{n^2}}$

The analogy is seen to hold at every step and to be present in the final values for a; and if a had been obtained directly by elimination of d or r from (1) and (2), the steps in the two processes would equally have followed the analogy. See also p. 140.

Analogies must not be pressed too far; but they have a use particularly as an aid to memory or to supply a hint. Thus, if a pupil could solve the above problem in A. P. the analogy would enable him to solve the corresponding problem in G. P.

We shall have later an analogy between mechanics and geometry (pp. 258–9). A statical analogy for a geometrical problem supplies the hint for generalizing the formula $a^2 = b^2 + c^2 - 2bc \cos C$ to a polygon (p. 216). Reference to the method will show that it is essentially the method of finding the resultant of a number of forces by, first, resolving in two directions and, then, compounding the resultants of the two sets of components.

This may be regarded as a case of vector addition, which can also be used to sum trigonometrical series.

Thus, to sum n terms of

$$\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \ldots$$

where $n\beta = 360°$, i.e. where β is the exterior angle of a regular polygon of n sides:

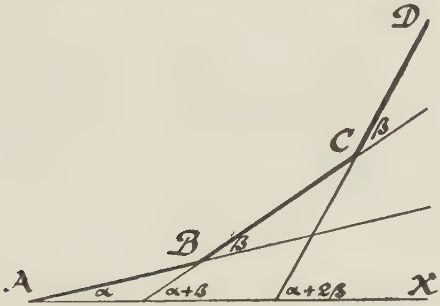


FIG. 140

Let ABCD . . . be the regular n-gon with sides of unit length; let AB make an angle α with AX.

Then $\quad \overline{AB} + \overline{BC} + \ldots$ to n terms $= 0$,

and therefore the sum of the projections of the sides along AX = 0.

i.e. $\quad \cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \ldots$ to n terms $= 0$.

This is the simplest case; but the sum of any number of terms of any similar series of cosines or sines can be summed in this way.

Or again, trigonometry may come to the help of algebra.

Thus to solve

$$bz + cy = a,$$
$$cx + az = b,$$
$$ay + bx = c,$$

we see that if a, b, c are the sides of a triangle and x, y, z are cos A, cos B, cos C, then the equations are trigonometrical identities. Now cos A, cos B, cos C cannot be greater than 1 or less than -1, whereas there need be no such restriction for x, y or z. Even so, we should expect that the solution for x would have the same form as cos A when expressed in terms of a, b, and c, viz. $\dfrac{b^2 + c^2 - a^2}{2bc}$, and this is so.

We may note, in passing, that the restriction $\cos A \not> 1$ or $\not< -1$ is related to the restriction that two sides of a triangle are together greater than the third, i.e. a restriction in the values of one of the magnitudes a, b and c as compared with the sum of the other two. In the group of algebraical equations there is no restriction of magnitude for a, b, and c; and if a, b and c are not so restricted, the values of x, y, z may be outside the limits -1 and 1, in which case the solution would correspond to values of cos A, etc., for imaginary triangles (*see* p. 207).

In the problem "Given a straight line XY and two points A, B in its plane, to find a point P in XY so that AP + PB is a minimum," there is an analogy between optics and geometry. (*See* also p. 286.)

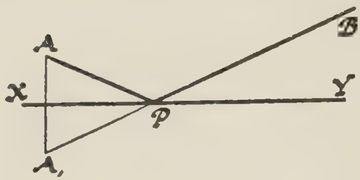


FIG. 141

A ray of light emanating from A and reaching B after reflection at a mirror XY traverses a path such that $\angle$ APX = $\angle$ BPY. This geometrical fact is the essential point of the solution of the geometrical problem. But the analogy takes us farther: in the case of the mirror, an observer at B sees the reflection of A as being apparently at A_1, a point symmetrical with A with respect to XY. And the geometrical figure that shows the positions of A_1, B, and A gives the clue to the proof of the geometrical problem.

There is an analogy between the sides of a triangle and the plane angles that form a trihedral angle.

"Two plane angles of a trihedral angle are together greater than the third" is analogous to, and is proved by, "Two sides of a triangle are together greater than the third." And just as the latter can be extended to a polygon so the former can be extended to a polyhedral angle.

Analogies and other correspondences may, as we said above, serve as mnemonics. Thus if AB is a thin, uniform rod, its centre of gravity G_2 bisects it.

If ABC is a thin, triangular lamina, its centre of gravity G_3 is a point of trisection (the nearest to G_2) of CG_2.

If ABCD is a solid tetrahedron, its C.G. is a point of quadrisection (the nearest to G_3) of DG_3.

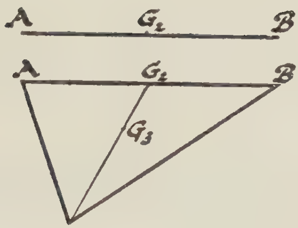


FIG. 142

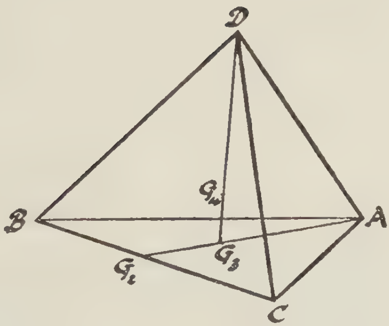


FIG. 143

For the 2-pointed figure AB, bisection gives the C.G.;
3- do. ABC, trisection gives the C.G.;
4- do. ABCD, quadrisection gives the C.G.

This is a similar case: The area of a triangle (2 dimensions) = $\frac{1}{2}$ area of parallelogram on the same base and having the same altitude; the volume of a pyramid (3 dimensions) = $\frac{1}{3}$ volume of the prism on the same base and having the same altitude.

Again the formula

$$\tan(A + B + C + \ldots) = \frac{t_1 - t_3 + t_5 - \ldots}{1 - t_2 + t_4 - \ldots},$$

(where t_r is the sum of all products of tan A, tan B, . . ., taken r at a time), is in a form easy to remember; if the connexion between tan θ and cot θ, viz. that one is the reciprocal of the other, is borne in mind, the formula serves also as a mnemonic for

$$\cot(A + B + C \ldots) = \frac{1 - c_2 + c_4 - \ldots}{c_1 - c_3 + \ldots},$$

where c_r is the sum of products of cot A, cot B, . . ., taken r at a time.

Most formulæ which present any feature of symmetry contain in their form an aid to memory. Nothing could be easier to remember than $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$; but such a formula as $\tan\frac{A - B}{2} = \frac{a - b}{a + b} \cdot \frac{\cot C}{2}$ misses this property, although the penultimate step in the proof has it. It might therefore be better to use the more easily remembered form.

$$\frac{\tan\frac{A - B}{2}}{\tan\frac{A + B}{2}} = \frac{a - b}{a + b}.$$

CHAPTER XI

DEGREE

In division there arises the question of the nature as well as the numerical magnitude of quantities—a question of fundamental importance in some branches of applied mathematics.

In quotition, " to divide 20 by 5 " appears under the guise " how many times are 5 nuts contained in 20 nuts ? " In division it appears in the form " divide 20 nuts into 5 heaps or among 5 boys."

In abstract arithmetic, as in algebra, we do not concern ourselves with boys, heaps, nuts, etc., but work with symbols such as the numbers 20 and 5 according to certain laws of manipulation ; in concrete examples, as in geometry and mechanics, the quantities we deal with involve a unit or denomination as well as a number, e.g., 5 ft., 6 secs., etc., 8 units of force, *m* units of momentum. The quantities may be written as products of the number and a unit, and the commutative law of multiplication and the rule of cancelling in division may be applied both to the number and to the unit in such a way that when a calculation is finished the right unit or denomination emerges in the answer. The importance of this in mechanics is usually made to appear in the teaching of the subject, and of course it appears in mensuration ; but there is no reason why it should not find a place more frequently in arithmetic.

Thus, writing 20 nuts as 20×1 nut, the quotition sum appears : 20 nuts divided by 5 nuts,

i.e. $$\frac{20 \times 1 \text{ nut}}{5 \times 1 \text{ nut}}.$$

Cancel the 1 nut and 5 and the answer appears as the pure number 4, which is independent of units and denominations. This is the converse of :—4 times 5 nuts, i.e. $4 \times 5 \times 1$ nut $= 20 \times 1$ nut $= 20$ nuts.

In division we have 20 nuts divided *among* 5 boys,

i.e. $$\frac{20 \times 1 \text{ nut}}{5 \times 1 \text{ boy}}.$$

The 5 cancels, but the 1 boy and 1 nut do not, and we get

$$\frac{4 \text{ nuts}}{1 \text{ boy}},$$

i.e. 4 nuts per boy. In this answer units are involved.

This is the converse of "If 5 boys have nuts at the rate of 4 nuts per boy, how many nuts have they in all?" They have

$$\frac{4 \text{ nuts}}{1 \text{ boy}} \times 5 \text{ boys},$$

i.e.
$$\frac{4 \times 1 \text{ nut} \times 5 \times 1 \text{ boy}}{1 \text{ boy}}.$$

Here "1 boy" cancels and the answer is

$$\begin{aligned} 4 \times 1 \text{ nut} \times 5 = 4 \times 5 \times 1 \text{ nut} &= 20 \times 1 \text{ nut} \\ &= 20 \text{ nuts}. \end{aligned}$$

The difference in the character of the answers in quotition and division is the difference between ratio and rate. Ratio involves no units, and we speak of the ratio of two quantities of the same denomination; rate involves units, and we speak of the rate of so many units of one thing per one unit of the other.

Multiplication of quantities is only possible where the product of the units involved is susceptible to interpretation: thus 2 ft. $\times$ 3 ft. $=$ 2 $\times$ 1 ft. $\times$ 3 $\times$ 1 ft. $=$ 2 $\times$ 3 $\times$ 1 ft. $\times$ 1 ft. $=$ 6 times the product of 1 ft. and 1 ft.

The product of 1 ft. and 1 ft. is interpreted as a sq. ft., and so 2 ft. $\times$ 3 ft. gives 6 sq. ft. But 2 lbs. $\times$ 3 lbs. is impossible, since 1 lb. $\times$ 1 lb. is a product for which there is no interpretation.

The division of 10 cu. ft. by 2 lin. ft. can be worked as

$$\begin{aligned} \frac{10 \times 1 \text{ ft.} \times 1 \text{ ft.} \times 1 \text{ ft.}}{2 \times 1 \text{ ft.}} &= 5 \times 1 \text{ ft.} \times 1 \text{ ft.} \\ &= 5 \text{ sq. ft.}, \end{aligned}$$

and interpreted as giving 5 sq. ft. for the base of a right prism whose volume is 10 cu. ft. and height 2 ft. (It can also be interpreted as 5 cu. ft. of volume per 1 ft. of height.)

Our experience leads us to state that concrete multiplication of quantities can only be performed in a limited number of cases, the quantities being restricted to numbers and geometrical measurements, the products in the latter case not proceeding beyond cubic measure.

Greek mathematics of the Classical period (say from 500–200 B.C.) only recognized products of geometrical measurements thus restricted, and of quantities that could be represented by them. Thus the algebraical identity $(a + b)^2 \equiv a^2 + 2ab + b^2$ appears as the geometrical identity of areas of rectangles; the solution of a quadratic as the conditions under which two areas may be equal.

All equations of measured quantities must be composed of terms of the same kind, or, as we say, "of the same degree." The word homogeneous (Gk. ὁμός, same and γένος, kind) is used for this. We use "degree" in this way: a linear measurement is said to be of the 1st degree; it involves linear measurement once. An area is said to be of the 2nd degree; it involves the product of two linear measurements. A volume is said to be of the

3rd degree, as it involves the product of three linear measurements. A pure number, being independent of dimensions, is of "no degree." In algebra, by analogy we say that single symbols a, b, c, x, y, z are of the 1st degree, and so are sums and differences of multiples of these as $2a + 3x$, $4b - 7y$, and these can be represented by lengths of straight lines; products of two, a^2, xy, etc., are of the 2nd degree, and can be represented by areas; products of three, a^3, b^2c, xyz are of the 3rd degree, and can be represented by volumes. Here geometry stops, but algebraical formulæ involve terms of higher degree. Thus in Heron's formula $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$, s, $s-a$, $s-b$, $s-c$ are the lengths of lines and their product is of the 4th degree, which would have been unthinkable for the earlier Greek mathematicians.

Again, whereas in geometry there must be homogeneity, algebra is not so restricted. Taking Heron's formula again, just as in algebra $\sqrt{n^4} = n^2$, i.e. is of the 2nd degree, so $\sqrt{s(s-a)(s-b)(s-c)}$ is of the 2nd degree, which is homogeneous with $\triangle$, an area. The algebraical expression $x^2 + 2x + 1$ is not homogeneous and can only refer to geometry if we assume that it is a particular case of $x^2 + 2cx + c^2$, where c is taken as 1 linear unit.

Consider the formula for a $\triangle$, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$. Here a is a line, i.e., of the 1st degree, sin A is a ratio and therefore of no degree. Therefore, $\dfrac{a}{\sin A}$ is linear and so the measurement of some line—the line is the diameter of the circum-circle.

In the Cartesian equation of a straight line

$$x \cos \alpha + y \sin \alpha = p,$$

x and y, the measurements of co-ordinates are of the 1st degree, $\cos \alpha$ and $\sin \alpha$ being ratios are of no degree; the left-hand side being homogeneous of the 1st degree, p must be of the 1st degree and so the measurement of a line. It is the length of the perpendicular from the origin to the line. If the form of the equation is

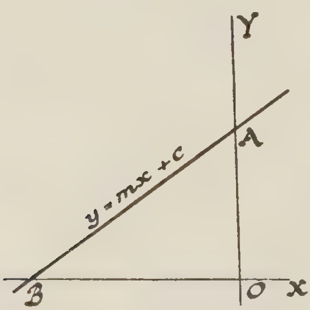


FIG. 144

taken as $y = mx + c$, then for homogeneity, m must be of no degree and c of the 1st degree; m is tan ABO, and c is the length AO (Fig. 144).

In the equations of motion of falling bodies,

$$s = ut + \tfrac{1}{2}gt^2,$$

let L, T represent a unit of length and a unit of time, then the dimensions of u are L/T, since it is the rate at which a length is traversed per unit of time; the dimensions of g are $\frac{L}{T^2}$, since it is the rate at which velocity increases per unit of time. Therefore, the dimensions of ut are $\frac{L}{T} \times T$, i.e. L; and the dimensions of gt^2 are $\frac{L}{T^2} \times T^2$, i.e. L; and the equation is homogeneous of the first degree in L.

Force is measured by the velocity produced in a unit mass in a unit of time. Its dimensions then are $\frac{MV}{T}$, i.e. $\frac{ML}{T^2}$.

Now Newton's law of attraction gives the formula for the force of attraction between two masses m_1, m_2 at a distance d, as $\frac{m_1m_2}{d^2}$, of which the dimensions are $\frac{M^2}{L^2}$, and these are not the dimensions of a force. But here the law gives not an equation but a variation, i.e.

$$\text{Force} \propto \frac{m_1m_2}{d^2};$$

and if we introduce a constant c so that

$$\text{Force} = \frac{c \,.\, m_1m_2}{d^2},$$

c is not a pure number, but a measured quantity involving the units $\frac{L^3}{MT^2}$.

The law can be stated thus:

$$\text{Force of attraction : unit force} :: \frac{m_1m_2}{d^2} : \frac{1\text{ lb} \times 1\text{ lb.}}{1\text{ sq. ft.}},$$

or $$\text{Force} = \text{unit force} \times \left(\frac{m_1}{1\text{ lb.}}\right) \left(\frac{m_2}{1\text{ lb.}}\right) \div \left(\frac{d^2}{1\text{ sq. ft.}}\right),$$

i.e. unit force multiplied and divided by a number of ratios.

Although algebra deals with abstract quantities and the question of units or dimensions may be immaterial, the ideas of degree and homogeneity are frequently applicable.

Thus, just as

$$\text{Volume} = \text{length} \times \text{length} \times \text{length or}$$
$$= \text{surface} \times \text{length}$$

so a 3rd degree expression may factorize into either three factors of the 1st degree or one of the 2nd and one of the 1st. No other

way is possible. A homogeneous expression must factorize into homogeneous factors.

Thus, $a^2 - ab - d^2 + bd$ is a homogeneous expression of the 2nd degree; it can only factorize into two of the 1st degree, both homogeneous, and

$$\begin{aligned} a^2 - ab - d^2 + bd &= a(a - b) - d(d - b) \\ &= (a - d)(a - b)(d - b) \end{aligned}$$

is shown by the degree of the result to be absurd.

Again,

$$\begin{aligned} x^2 - p^2 + 3(x - p) &= (x - p)(x + p) + 3(x - p) \\ &= (x - p)(x + p + 3x - 3p) \\ &= (x - p)(4x - 2p) \end{aligned}$$

is equally absurd, since a heterogeneous expression—an expression in which the terms are of different degree (Gk. ἕτερος = different) —has resolved into homogeneous factors. These results are indeed just as absurd as to express area in cubic measure or to attempt to add area and length.

To factorize

$$(a^2 - 3ac + 2c^2)\,x^2 + (5c^2 - a^2)\,x - 2(a^2 - c^2) \quad \ldots\ldots (A)$$

The expression is homogeneous in a and c; and, as a function of x, it is of the quadratic type. Either it will not factorize at all or the factors will be binomial expressions of the 1st degree in x with coefficients homogeneous in a and c. If in one factor the coefficients are homogeneous of the 2nd degree in a and c, in the other they will be of no degree. If in one factor the coefficients are homogeneous of the 1st degree in a and c, they will be homogeneous of the 1st degree in the other. Thus we may try

$$[(a^2 - 3ac + 2c^2)\,x \pm (a^2 - c^2)]\,[x \mp 2] \quad \ldots\ldots (I)$$
$$[(a^2 - 3ac + 2c^2)\,x \pm 2(a^2 - c^2)]\,[x \mp 1] \quad \ldots\ldots (II)$$
$$[(a - 2c)\,x \pm (a - c)]\,[(a - c)\,x \mp 2\,(a + c)] \quad \ldots\ldots (III)$$
$$[(a - 2c)\,x \pm 2(a - c)]\,[(a - c)\,x \mp (a + c)] \quad \ldots\ldots (IV)$$
$$[(a - 2c)\,x \pm (a + c)]\,[(a - c)\,x \mp 2(a - c)] \quad \ldots\ldots (V)$$
$$[(a - 2c)\,x \pm 2(a + c)]\,[(a - c)\,x \mp (a - c)] \quad \ldots\ldots (VI)$$

Now, in (I), (II), (V) and (VI) $a - c$ appears as a factor; but as it is not a factor of the original expression, these forms are impossible. There remain only (III) and (IV) to consider, and by putting $a = 0$ in (IV) we get 2 a factor, whereas by putting $a = 0$ in (A) we do not. Therefore (IV) may be dismissed. By putting $a = 0$ in (III), or by finding the coefficient of x in the product, we decide between the alternatives in sign and get

$$[(a - 2c)\,x + (a - c)]\,[(a - c)\,x - 2\,(a + c)].$$

We see here, as we see in quadratic forms generally, that a regular drop in the degree in the quadratic is accompanied by the same regular drop in degree in the factors. Thus in $x^2 + 7x + 12 \equiv (x + 3)(x + 4)$:—both in the quadratic function and in the factors, the degree of the terms drops regularly by 1. Again, in $\left(x + 2 + \frac{1}{x}\right)^2 = x^2 + 4x + 6 + \frac{4}{x} + \frac{1}{x^2}$ the same behaviour is

exhibited, and here we are led to talk of expressions of -1 degree, or 1st degree down, and -2 degree, or 2nd degree down, in the case of $\frac{1}{x}$ $\frac{1}{x^2}$. And if D_2 represents a term of the 2nd degree, D_1 of the 1st, then D_0 represents a term of no degree, such as 2 or 6; and D_{-1} represents a term of the "1st degree down," as $\frac{1}{x}$; and D_{-2} a term of "the 2nd degree down," as $\frac{1}{x^2}$.

In the same way, in $(x^2 - y^2)^2 - 2(x^2 + y^2) + 1$, where the degree falls regularly by 2 (the expression being of the form $D_4 + D_2 + D_0$), we might expect the factors to exhibit the same descent.

The 4th degree term may factorize into (1) two factors of the 2nd degree, (2) one of the 3rd and one of the 1st; but the 2nd and 3rd degree terms may factorize further. Now as a 1st degree term, being of the form $D_1 + D_0$, does not exhibit this fall, we shall try (1), i.e. two terms of the form $D_2 + D_0$; to give this $(x^2 - y^2)^2$ will factorize only as $(x^2 - y^2)\ (x^2 - y^2)$ and $(x - y)^2\ (x + y)^2$. This first would give as factors

$$[(x^2 - y^2) - 1]\ [(x^2 - y^2) - 1],$$

which is not right. The second would give

$$[(x - y)^2 - 1]\ [(x + y)^2 - 1],$$

which is.

But by further factorization this reduces to

$$\{(x - y) + 1\}\ \{(x - y) - 1\}\ \{(x + y) + 1\}\ \{(x + y) - 1\},$$

i.e. 4 factors of the form $D_1 + D_0$. We saw that when the degree of the terms of an expression descended by 2, it was possible to reduce it to factors in which the degree of the terms descended by 2. But it was not in this case impossible to reduce to factors where the degree of the terms descended by 1. In other cases it might be. At any rate, it is easier to proceed through the drop of 2 in degree; but it leads to generality to consider that in all expressions of ascending or descending degree the difference of degree from term to term is 1, and to supply terms of missing degree with zero coefficients when there is any greater difference. Thus the expression $(x^2 - y^2)^2 - 2(x^2 + y^2) + 1$ can be regarded as of the form $D_4 + D_3 + D_2 + D_1 + D_0$ where the coefficients in D_3, D_1 are zero. Just as in $x^2 - 1 = (x + 1)\ (x - 1)$ the expression $x^2 - 1$ is really $x^2 + (x - x) - 1$, and so of the form $D_2 + D_1 + D_0$.)

An algebraical function is said to be of the nth degree in x (n being a positive integer), when it is made up of terms of the form $1, x, x^2, \ldots$ up to x^n, the terms having any numerical coefficients.

Such an expression has n factors of the form $x + a + b\sqrt{-1}$,

where a and b are any real numbers. (In elementary algebra we consider only linear factors in which $b = 0$.)

It follows that an equation of the nth degree in x, i.e. one in which a function of the nth degree is equated to zero, there will be n solutions of the form $p + q\sqrt{-1}$.

It also follows that if a function of the nth degree be graphed, a straight line may cut the graph in n points. Thus, if we draw the graph of

$$y = ax^n + bx^{n-1} + cx^{n-2} + \ldots + kx + l,$$

the line $y = mx + c$ will meet it where

$$mx + c = ax^n + bx^{n-1} + cx^{n-2} + \ldots + kx + l,$$

and this is an equation of the nth degree, having n solutions for x, viz: the abscissæ of the points of intersection. These values of x are of the form $p + q\sqrt{-1}$, and if in all cases $q = 0$, then there are n real points. Other cases will be dealt with in a later chapter; here we will give a few illustrations of cases in which $q = 0$. Thus, if we plot the parabola $y = x^2$, the simplest 2nd degree function,

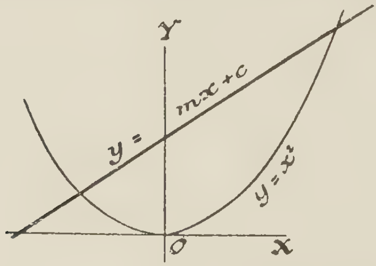


FIG. 145

lines can be drawn to cut it in 2 and no more than 2 points; it has one bend.

The cubical parabola, Fig. 146 (1), $y = (x - 1)\,x\,(x + 1)$ can be cut in 3 and no more than 3 points; it has two bends.

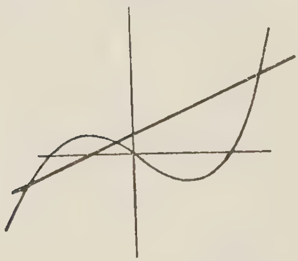


FIG. 146 (1)

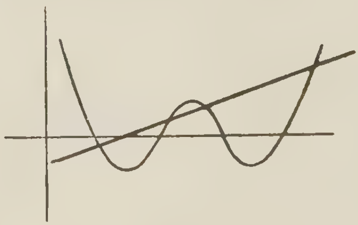


FIG. 146 (2)

$y = (x - 1)(x - 2)(x - 3)(x - 4)$, Fig. 146 (2), being of the 4th degree, can be cut in 4 and no more; it has 3 bends. And so on.

And if y is a general function of the nth degree, straight lines may be drawn to cut the graph in n points, and there will be $n - 1$ bends, or turning-points.

These are explicit functions, that is to say the y is not "mixed up" with the function of x; but when y is an implicit function of x, as in the equation $x^3 + y^3 = 3axy$ (where x and y are not easily separable), it is equally true that a straight line may meet

FIG. 147

FIG. 148

the graph of an nth degree function in n and no more than n points. One or two simple illustrations are given.

The cissoid of Diocles (Fig. 147) is $x^3 = (2a - x)y^2$. (*See* p. 41.)

If $mx + c$ is substituted for y, an equation of the 3rd degree in x is obtained. It can be seen in the diagram that a straight line can be drawn to cut the curve in 3 points.

The lemniscate (Fig. 148) $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ is of the 4th degree, and the degree is not affected by the linear substitution $mx + c$ for y. A straight line may therefore meet it

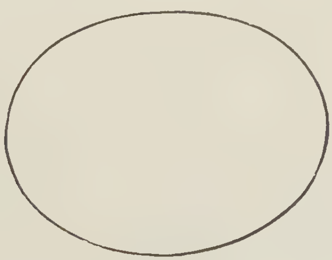


FIG. 149

in 4 points, which agrees with what we can see from drawing the figure.

There are cases where it is not possible to draw any line to cut a curve of the nth degree in n points, e.g. the curve $\frac{x^4}{a^4} + \frac{y^4}{b^4} = 1$ (Fig. 149) is not distinguishable in appearance from an ellipse,

and no straight line will cut it in more than 2 real points. The complete explanation of these is beyond the scope of this book; but it still remains true that (1) a straight line cannot cut a curve of the nth degree in more than n points, (2) if a straight line cuts a curve in n points the degree of its equation must be at least the nth. And even this amount of certainty is useful. In particular, straight lines can meet other straight lines in only one point, and the equation of the straight line is of the 1st degree.

Straight lines may meet conics in 2 and no more points; all conics are of the 2nd degree with the general equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

For proof see p. 71. The converse and complementary proposition is also true, viz: that every equation of the 2nd degree is a conic.

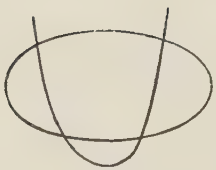


FIG. 150 (1)

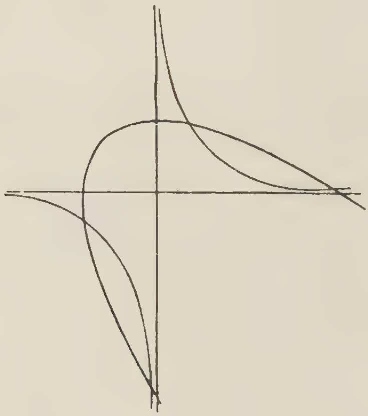


FIG. 150 (2)

It may now be asked in how many points a curve of the 2nd degree intersects a curve of the 2nd or 3rd degree. Two conics may meet in 4 points; thus in Fig. 150 (1) a parabola and an ellipse, and in Fig. 150 (2) a parabola and a hyperbola, are so placed as to give 4 points of intersection. There is one exception: two circles intersect in only 2 real points (*see* page 43).

The cubical parabola and a circle may meet in 6 points (Fig. 151). In these cases the possible number of points of intersection is the product of the degrees of the equation; and it is generally true that a curve of the nth degree may intersect a curve of the mth in mn points.

Correspondingly, the number of solutions for a pair of simultaneous equations of the nth and mth degrees is mn.

FIG. 151

The following argument, which can be extended, is confirmatory. Consider

$$ax^2 + hxy + by^2 + gx + fy + c = 0 \quad \ldots\ldots\ldots \quad \text{(I)}$$

and

$$lx^3 + mx^2y + ny^2 + px + d = 0 \quad \ldots\ldots\ldots \quad \text{(II)}$$

(I) is a general equation of the 2nd degree in x and y, (II) is some equation of the 3rd degree.

(I) can be regarded as a quadratic in y, thus :

$$by^2 + (hx + f)y + (ax^2 + gx + c) = 0,$$

and may by factorization reduce to

$$(y - m_1x - c_1)(y - m_2x - c_2) = 0,$$

i.e. there are two values in terms of x for y, viz., $m_1x + c_1$ and $m_2x + c_2$. Each of these can be substituted severally in (II), giving an equation of the 3rd degree in x, for which there are three solutions ; i.e. for each linear relation obtained from (I) there are three values of x, i.e. for the two relations there are 2×3 values of x.

Similarly, in a group of simultaneous equations of three unknowns of degree l, m, and n the number of solutions is the product lmn.

The degree of an equation must not be confused with the degree of a magnitude. We have seen that an identity of the 2nd degree can be demonstrated and an equation of the 2nd degree in one unknown can be solved by the identity and equation of areas ; but we have also seen that an equation of the 2nd degree in x and y is represented not by an area but a line. Although a surface is an area, the equation of a surface is not necessarily of the 2nd degree in x and y. It is, in general, an equation in three unknowns. A plane is represented by an equation of the form $ax + by + cz = d$, i.e. a 1st degree equation in 3-dimensional units, just as $y = mx + c$, the equation of a straight line, is of the 1st degree in 2-dimensional units. Again, a quadric is of the 2nd degree in x, y, z, just as a conic is of the 2nd degree in x and y ; thus, a sphere is $x^2 + y^2 + z^2 = r^2$ just as a circle is $x^2 + y^2 = r^2$.

The fact is that in algebraic geometry the equation represents

a relation between the co-ordinates x and y (or in 3 dimensions between x, y, and z) expressed as a rational equation.

Sines and cosines are geometrically of no degree, but it is sometimes convenient to regard them algebraically as magnitudes of the 1st degree, like x or y, independently of their geometrical meaning.

Then $\sin(A + B)$, which $= \sin A \cos B + \cos A \sin B$, is of the 2nd degree, as is the particular case $\sin 2A = 2 \sin A \cos A$; $\cos 2A$ in the form $\cos^2 A - \sin^2 A$ is of the 2nd degree and homogeneous; it loses homogeneity when written $2 \cos^2 A - 1$, but retains, at any rate, one term of its characteristic degree.

$\sin 3A = 3 \sin A - 4 \sin^3 A$ is also derived from the expression $3 \sin A \cos^2 A - \sin^3 A$ by using the substitution $\sin^2 A + \cos^2 A = 1$. It has lost homogeneity by the substitution, but has retained a term of its characteristic degree.

$\sin nA$ and $\cos nA$ can be expanded as homogeneous functions of the nth degree in $\sin A$ and $\cos A$.

With the above assumption $\tan A = \dfrac{\sin A}{\cos A}$ is a homogeneous function of no degree in sine and cosine, and this is often used in trigonometrical analysis.

The equation $a \sin \theta + b \cos \theta = c$ can be made homogeneous by the following substitutions:

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2},$$

$$\cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2},$$

$$1 = \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2},$$

and will be written

$$2a \sin \frac{\theta}{2} \cos \frac{\theta}{2} + b\left(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}\right) = c\left(\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}\right).$$

After division by $\cos^2 \frac{\theta}{2}$ it becomes

$$2a \tan \frac{\theta}{2} + b\left(1 - \tan^2 \frac{\theta}{2}\right) = c\left(1 + \tan^2 \frac{\theta}{2}\right),$$

a quadratic in $\tan \frac{\theta}{2}$, which solves the original equation.

THE FOURTH DIMENSION

The question of 4-dimensional space is naturally connected with the question of degree; and some remarks on it may be made here.

Living on a 3-dimensional world we have some conception of

space of 2 dimensions or of 1 dimension. Indeed, our geometrical investigations are mostly 2-dimensional. Plane figures can be accurately represented on our paper or blackboard; a solid figure can be pictured but not accurately represented; in perspective or other representations of solids on a plane there is always distortion of the magnitudes of some lines or angles, or both; and we investigate the solid through a consideration of various plane aspects. Three-dimensional space being difficult and intricate, 4-dimensional space is outside our powers of perception.

FIG. 152

Consider the nature of 1-dimensional space, which we will call Lineland. It will have length but neither breadth nor thickness. Let AB, CD, and EF be Linelanders living in XY. They can move towards X or towards Y, but in no other way. Assuming that they can perceive a "point," CD can make the acquaintance of only two neighbours, but he will know nothing about their length and he can never see their remote extremities A and F.

Again AB can see only the extremity C of CD, and EF can see only D. But if CD could be moved out of XY in a surface containing it, he could be turned round and replaced so that AB could see D, and EF could see C (Fig. 153).

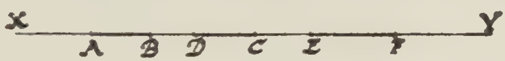


FIG. 153

Or again he could be put back into XY so that he could make the acquaintance of EF's other neighbour and thus extend his visiting list. As the Linelanders are not conscious of the 2nd dimension, these enlargements of their narrow world will not occur to them.

FIG. 154

But if some day they found that one of these changes had taken place in their positions, a particularly imaginative Linelander might supply an explanation by the hypothesis of a 2nd dimension.

In 2-dimensional space, about which an interesting story, FLATLAND, has been written, the inhabitants would be geometrical figures having length and breadth, but no thickness. Here, supposing that they could perceive "lines," each could become acquainted with every other Flatlander. But they could not play leap-frog or turn somersaults. Their upper (or outward) surfaces would

always be upper (or outer), like flat-fish on a sea-bottom, never downward (or inner). To change upper surface to lower, it would be necessary for the Flatlander to make an excursion into 3-dimensional space and be turned round; in doing so his right hand would become left and his left hand right.

In H. G. Wells's "Plattner Story" the hero, after a temporary disappearance, reappears with this sort of displacement—his heart is on his right side and so on, a change which takes place during a visit to 4-dimensional space.

Again consider the phenomenon in Lineland of a circular lamina Q passing across XY between AB and CD.

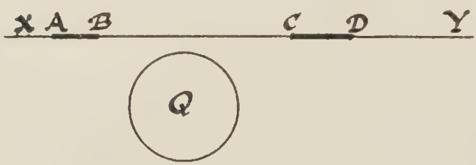


Fig. 155

As it makes contact with XY, AB and CD become aware of the interposition of a new neighbour; as Q proceeds, this neighbour first appears to approach the Linelanders, then to retreat again, and finally to disappear as mysteriously as it had come.

So in the story Flatland a sphere passes through, its visible section makes a mysterious appearance, grows larger somewhat alarmingly, but diminishes again, and finally disappears. It would be open to us by analogy to explain phenomena of objects which appear and disappear in our world as possible visits of a being with 4-dimensional attributes into the 3-dimensional world we inhabit, a world that may be only one aspect of a 4-dimensional space which unknown to ourselves we form and live in. Finally, just as the Linelanders would be unable to say whether XY was straight or curved, extending to infinity in both directions or a continuous closed line like a large circle, and just as the Flatlander would be unable to say whether his world was plane or curved, extending to infinity, or forming a continuous surface without boundaries like the surface of a large sphere, so we could postulate similar difficulties concerning the nature of our space. The Linelander and Flatlander would have no perception of straightness or planeness respectively, but their mathematicians could develop an algebraic geometry dealing with these conceptions, and our mathematicians do for some purposes postulate a 4-dimensional space, in which a point has four co-ordinates, x, y, z, and u; develop its properties by algebraical analysis, and regard our space as a special aspect in which u is taken to be 0.

CHAPTER XII

CONTINUITY

SUPPOSE pennies of thickness p inches to be fed into the vertical cylinder of an automatic machine at the rate of 1 per minute. After t minutes the height of the column of pennies will be pt inches, providing that t is an integer. If t is not an integer, let $\mathrm{I}(t)$ represent its integral part; then the height is $p \times \mathrm{I}(t)$.

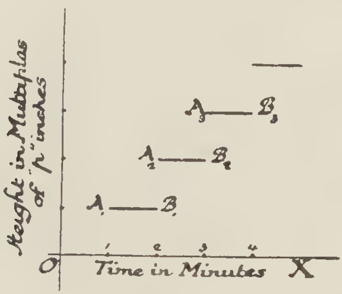


FIG. 156

The graph of the height as a function of the t me is shown in Fig. 156. It consists of O1 (along OX), A_1B_1, A_2B_2, etc., i.e. of a set of steps of equal line-segments parallel to OX. The height does not vary as the time; for each whole minute it remains unchanged, at the end of each minute it is increased, but by the same amount in each minute.

The end-points O, A_1, A_2, A_3, . . . are on a straight line, the graph of the function pt. The end-points B_1, B_2, B_3, . . . are also on a straight line, parallel to $OA_1A_2A_3$ . . ., the graph of $p(t - 1)$.

Now, suppose that, instead of pennies at 1 per minute, disks of $\frac{1}{60}\,p$ inches in thickness are fed in at 1 per second. Then the height after s seconds (s being an integer) would be $\dfrac{ps}{60}$ inches. And for any time s seconds, taking $\mathrm{I}(s)$ to be the integral part of s, the height would be $\dfrac{p}{60} \times \mathrm{I}(s)$.

The graph in this case would consist of 60 times as many line-segments, each of $\frac{1}{60}$ the length of A_1B_1; thus from O to A_1 there would be 60 steps instead of one, and so on. Of these line-segments one set of end-points would lie along OA_1A_2 . . ., and the other set would lie on a straight line parallel to OA_1A_2, such that the distance from it would be $\frac{1}{60}$ the distance of $B_1B_2B_3$ from it.

If further subdivision of the thickness of the disks and of the time-interval were supposed to be made, the graph would consist of a still greater number of line-segments of still shorter length with the second set of end-points approaching more nearly to the straight line OA_1A_2; i.e. it would form a graph of approximately direct variation, but would never become a continuous straight line.

If, however, we imagine a liquid to enter a cylinder uniformly at a rate of p inches of height per minute, its height-time graph would be the continuous straight line OA_1A_2 . . ., the graph of the algebraic function $y = pt$.

Simple Interest.—Consider again the case of money invested at simple interest. The formula $I = \frac{P \times r \times t}{100}$, taking P and r as given, is a formula of direct variation for I and t, and its graph is a continuous straight line. In practice this is only approximately true, for

(1) the time of investment is not taken as infinitely divisible. The smallest practical unit of time in banking business is the day, so t (in days) must be taken as I(T), I (T − 1) or I(T + 1), where T is the exact time, in days, of investment; in the Post Office Savings' Bank the unit is one month, and t (in months) is taken as I(T), or I(T − 1), where T is the exact time of deposit in months.

(2) When I is found it may not be a whole number of pennies, in which case the nearest whole number is taken.

The graph of the Amount of money invested at simple interest, if the first consideration only were taken into account, would be of the type A_1B_1, A_2B_2, etc., in Fig. 156. But as the second consideration must also be taken into account, these line-segments will not be at exactly equal vertical distances. That is to say, it is only for the algebraical formula that the graph is continuous. In practice the Amount of money invested at simple interest gives a broken graph.

Compound Interest.—For the Amount of money invested at compound interest B_0A_0, B_1A_1, B_2A_2, . . . (Fig. 157) is the graph, where OB_0 is the principal, A_0B_1, A_1B_2, A_2B_3 the interest in successive years, and $A_0B_1 < A_1B_2 < A_2B_3$.

The formula, Amount $= P\left(1 + \frac{r}{100}\right)^t$, or $A = PR^t$ where $R = 1 + \frac{r}{100}$, has, for positive values of t, a continuous graph, $B_0B_1B_2 \ldots$

Here, too, the divisibility of the penny as well as the divisibility of time-units must be considered, and may be illustrated by the following question, "*If* 1d. *had been invested at* 5 *per cent per annum, compound interest, on the day of the Battle of Waterloo, what would have been its amount on June* 18, 1915*?*"

In banking practice it would have remained 1d. According to the formula $A = PR^t$, which permits the addition of any fraction of a 1d. however small, the Amount would have been slightly more than 10s. 11½d.

If £1,000 instead of 1d. had been invested, the Amount resulting from banking book-keeping would have been nearly that obtained

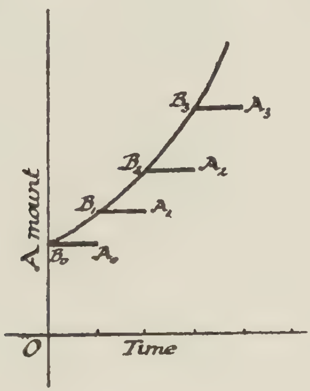


FIG. 157

from the formula, viz., £131,500 (to the nearest £100). But it is to be noticed that the answer to such a sum in compound interest as worked in school arithmetic may always differ a little from the book-keeping answer.

In actuarial work a formula which implies the infinite divisibility of time, i.e. which considers interest to be accruing continuously, is found convenient for many purposes.

Consider £1 invested at compound interest (1) for 1 year at 100 per cent per annum, (2) for n periods of $\frac{1}{n}$ years at $\frac{100}{n}$ per cent per period, where $n = 2, 3, 4, \ldots$ At simple interest in each case the Amount would be £2.

At compound interest we get the Amounts tabulated below.

First using 4-fig. tables we get

n	R	log R	n log R	A
2	1·5	·1761	·3522	2·250
4	1·25	·0969	·3876	2·441
5	1·2	·0792	·3960	2·489
8	1·125	·0502	·4016	2·522
10	1·1	·0414	·4140	2·594

These results for A cannot be regarded as more correct than log R is, i.e. to 3 figs. To proceed, we use 8-fig. tables.

n	R	log R	n log R	A
20	1·05	·0211 8930	·423 786	2·654
40	1·025	·0107 2387	·428 955	2·685
50	1·02	·0086 0017	·430 085	2·693
80	1·0125	·0053 9503	·431 602	2·702
100	1·01	·0043 2137	·432 137	2·705
200	1·005	·0021 6606	·433 21	2·7115
400	1·0025	·0010 8438	·433 75	2·7149
500	1·002	·0008 6772	·433 86	2·7156
800	1·00125	·0005 4253	·434 02	2·7166
1000	1·001	·0004 3408	·434 08	2·7170
2000	1·0005	·0002 1709	·434 1(8)	2·717
4000	1·00025	·0001 0856	·434 2(4)	2·718
5000	1·0002	·0000 8685	·424 2(5)	2·718
8000	1·000125	·0000 5429	·434 3(2)	2·718

In the last part of the table, where the interest is being added almost hourly, we are again reduced to 4-fig. accuracy in the results, and to 4 figures it seems as if the Amount is about £2·718, regarding time as infinitely divisible. A graph of A for different values of n will make this approach to a limit clearer than the table does.

We have found values of $\left(1 + \frac{1}{n}\right)^n$ for increasing values of n, and as n gets very large the value seems to be 2·718 to 4-fig. accuracy.

It can be shown by algebra that as $n \to \infty$

$$\left(1 + \frac{1}{n}\right)^n \longrightarrow 1 + 1/1! + 1/2! + 1/3! + \ldots \quad \text{(I)}$$

where 4! means $1 \times 2 \times 3 \times 4$.

Let us calculate this to, say, 8-fig. accuracy.

1	=	1·	
1 / 1!	=	1·	
1 / 2!	=	·5	dividing 1 / 1! by 2.
1 / 3!	=	·1666 66667	dividing 1 / 2! by 3.
1 / 4!	=	·0416 66667	dividing 1 / 3! by 4.
1 / 5!	=	·0083 33333	
1 / 6!	=	·0013 88889	
1 / 7!	=	·0001 98413	
1 / 8!	=	·0000 24802	
1 / 9!	=	·0000 02756	
1 / 10!	=	·0000 00276	
1 / 11!	=	·0000 00025	
1 / 12!	=	·0000 00002,	and we get no further digits in the 9th place of decimals.
		2·7182 81830	

That is, 2·71828183 is the value correct to 9 figures of the series (I) to which Euier assigned the symbol e. This series is of great importance in mathematics, especially in the theory of logarithms and the expansions of sin θ and cos θ.

We see, then, that if interest at 100 per cent is divided so that it is added as $\frac{100}{n}$ per cent per period for n periods, the increase is more than 171·8 per cent if n is made very large.

The algebraical theorem that establishes the series for e also proves that

$$\left(1+\frac{x}{n}\right)^n \to e^x \text{ as } n \to \infty,$$

i.e., putting $\frac{\alpha}{100}$ for x,

$$\left(1+\frac{\alpha}{100n}\right)^n \to e^{\alpha/100}.$$

Now if we consider money invested so that £1 becomes £$\left(1+\frac{r}{100}\right)$ in 1 year, and if we suppose interest at α per cent per annum to be added n times per annum so that

$$R \equiv 1+\frac{r}{100} = \left(1+\frac{\alpha}{100n}\right)^n \to e^{\alpha/100},$$

when $n \longrightarrow \infty$, then α per cent is the annual rate of interest, accruing continuously, which corresponds to a rate r per cent per annum, and α is called the effective rate. The graph $B_0B_1B_2$ . . . (Fig. 157) is then the graph of $A = P.e^{\alpha t/100}$.

As an example, the effective rate corresponding to 5 per cent per annum is given by

$$1{\cdot}05 = e^{\alpha/100},$$

$$\log(1{\cdot}05) = \frac{\alpha}{100}\log e,$$

$$\frac{\alpha}{100} = \frac{\log 1{\cdot}05}{\log e}$$

$$= \frac{{\cdot}02118930}{{\cdot}43429448} \times 100$$

$$= \text{about } 4{\cdot}87902.$$

This method of dealing with compound interest enables us to find an equitable value for the Amount for a period of time which would be represented by a fractional number of years. Thus,

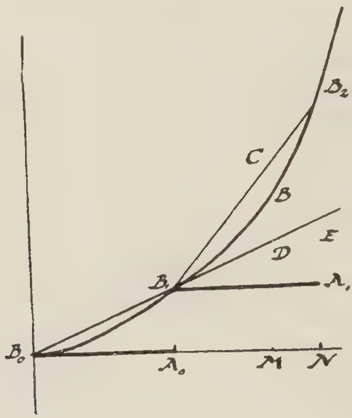


FIG. 158

suppose that £1 is invested for $1\frac{3}{4}$ years at 4 per cent per annum compound interest, the interest being added yearly:

In the first year £1 has amounted to £1·04.

At the end of the period the second addition of interest is not due. The interest equitably due might be taken (as it usually is in school) as $\frac{3}{4}$ of one year's interest on £1·04, i.e. £·0312, making an amount of £1·0712.

According to the formula $A = P(1{\cdot}04)^{1\frac{3}{4}}$, A is 1·071046, and at simple interest the Amount would be 1·07.

In the graph given in Fig. 158, let B_0A_0 and A_0N each represent one year and A_0M represent $\frac{3}{4}$ year. Let MDBC and NA_1EB_2 be the ordinates at M and N; A_0B_1 is the first year's, A_1B_2 the second year's interest reckoned as compound.

The smooth curve $B_0B_1BB_2$, being the graph of the formula for the Amount, should give true values of the Amount for a period

of any length. So MB represents the interest £·071046 for $1\frac{3}{4}$ years.

The straight line B_0B_1E is the graph of Amounts at simple interest, and MD is ·07. The straight lines B_0B_1, B_1B_2 give the Amounts at compound interest for whole numbers of years with fractional parts of a year's interest for a corresponding part of a year, i.e. at certain points, B_0, B_1, B_2, it obeys a compound interest law; for intermediate points the simple interest law is involved and MC is the 1·0712 obtained by the school method.

Where, however, a law holds continuously, a straight line or a smooth curve is the usual graph; and in plotting graphs of simple functions we usually assume that they are continuous. This, however, is not inevitably so, as a consideration of the graph of x^x, where fractional negative values are given to x, will show. Part of the graph will consist of a number of points as near to one another as we like to make them, but separated by very small gaps. But in this case the gaps are accounted for on algebraical, not practical, considerations.

Loci.—In geometry also there arises a point similar to the one we have been considering.

To determine the locus of points equidistant from two given points A and B, it is not sufficient to show that they all lie on the right bisector of AB; it is imperative to show also that all points on the right bisector are equidistant from A and B.

For to find the circum-centre of the $\triangle$ ABC, we show that it is the point of intersection of the locus of points equidistant from A and B and the locus of points equidistant from B and C. But if the loci are not continuous, the right bisectors of AB and BC might intersect at a point where there is a gap in one or both of the loci; and so no real circum-centre would be obtained; i.e. constructions which depend on the intersection of loci depend on continuity of line in the loci.

If the circumference of a circle is defined as the locus of coplanar points which are equidistant from a given point, we have not a clear statement that the circumference is a continuous line, and constructions with circles would not be satisfactory. Euclid's definition of the circle as a figure bounded by a line implies the continuity of the circumference. This is the static definition. The definition of the circumference as a line traced out by a moving point—the dynamic definition—also implies continuity. In practice straight lines and circles, being constructed by tracing-points, are continuous, but in Greek abstract geometry movement was avoided and the question of continuity became important.

In arithmetic, too, the question of continuity arises, e.g. th intervals between 0 and 1, 1 and 2 can be subdivided fractionally, and the results represented on a scale, as on a ruler. The point on the scale corresponding to any fraction can be constructed; but when all possible points are supposed to have been so con-

structed there remain others. Thus, if n is not a perfect square, $\sqrt{n}$ cannot be constructed by fractional division, but its position can be otherwise fixed. And in any case it can be shown to lie between two points, as near as we like to each other, that are obtainable by fractional division, thus $\sqrt{2}$ lies between 1 and 2; 1·4 and 1·5; 1·41 and 1·42; 1·414 and 1·415; 1·4142 and 1·4143, and so on.

$\sqrt[3]{2}$ cannot be fixed by Euclidean construction, but like $\sqrt{2}$ its value can be shown to lie between two points obtained by fractional division.

Fig. 159

Other points on the line not corresponding to fractions or surds correspond to other numbers, such as logs of numbers, sines, etc., of angles. Their places in the number scale can be assigned as lying between two points obtained by fractional division, the numbers corresponding to these points being obtained by calculation from series.

The innate feeling that continuity is inevitable has played its part in extending mathematical, and especially algebraical, knowledge. But though most intuitions act as safe guides, there are others which are misleading or at least misdirecting.

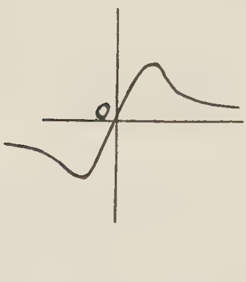


Fig. 160

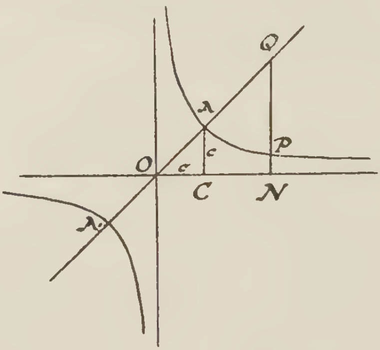


Fig. 161

Few beginners plot correctly at their first attempt the function $\frac{1}{x}$. They commonly draw it as in Fig. 160.

They grope after continuity, but get on to the wrong track.

Now $y = x$ is a continuous straight line passing through the origin. Let Q (Fig. 161) be any point on it and QPN the ordinate of Q, with P taken so that

$$PN \, . \, QN = c^2$$

Then P is on the line $y = \frac{c^2}{x}$ (this line meets the straight line OA where $y = x = \pm c$).

Now since $\text{PN} = \frac{c^2}{\text{QN}}$; for any value of QN there is one, and only one, corresponding value of PN, and our feeling that the locus of P is continuous is intensified by this relation; for if Q is conceived as moving continuously along its locus, it seems that P cannot do otherwise along its locus. But as Q passes through O from the first to the third quadrant, P goes from $+\infty$ to $-\infty$.

To satisfy continuity we must postulate some sort of mathematical contiguity for $+\infty$ and $-\infty$.

This behaviour of functions at infinity seems fairly general.

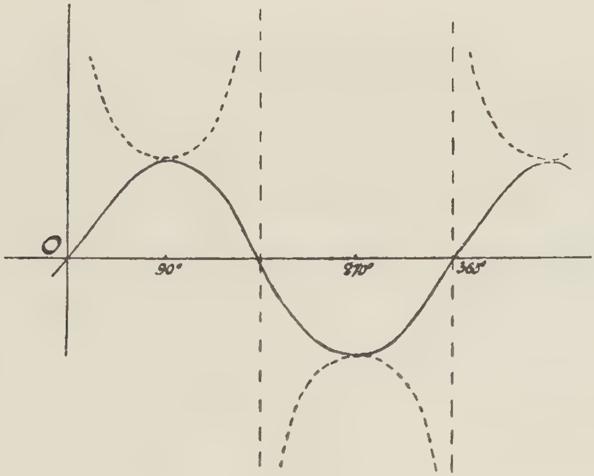


FIG. 162

Thus the sine graph, shown by the unbroken line in Fig. 162, is a continuous periodic curve. The graph of the reciprocal, the dotted line, exhibits the behaviour for $+\infty$ and $-\infty$ that we have noticed in $y = \frac{1}{x}$.

The graph of $y = \frac{1}{x^2}$ (Fig. 163), does not at first sight show the contiguity of $+\infty$ and $-\infty$ for values of y as x passes through O; but if we plot $y = \frac{1}{x^2 - c^2}$ we get a three-branched curve which

does (Fig. 164), and by making c smaller, and eventually zero, we come to $y = \frac{1}{x^2}$ as a special case. In the graph of $y = \frac{1}{x^2}$, then, we have to remember that one branch has, so to speak, been squeezed out.

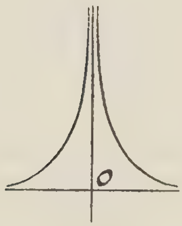


Fig. 163

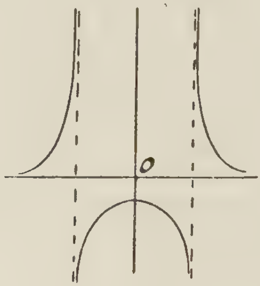


Fig. 164

The feeling for continuity shows itself in other ways, e.g. in continuity of argument.

In the method of mathematical induction, having proved that if a proposition is true for a value n of a variable, it is therefore true for the value $n + 1$, we can use this result to prove the universal truth of the proposition, providing we can prove it true in one special case.

In the case of various loci which appear to be arcs of a circle, this feeling leads us to consider the significance of the residual arc of the circumference. The unattached ends of the arc-locus are abhorrent and clamour for investigation.

Consider a $\triangle$ APB with AB fixed and $\angle$ APB given in magnitude. The locus of P is an arc of a circle of which AB is a chord.

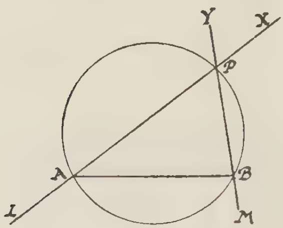


Fig. 165

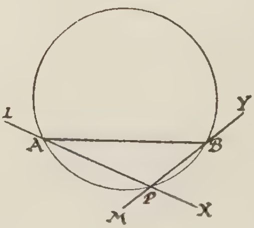


Fig. 166

If we state the proposition thus: to find the locus of the point of intersection P of two lines LPX and MPY which meet at a constant angle LPM, and always pass through fixed points A and B, we find that P may lie anywhere on the circumference of the circle.

For consider a position of P on the opposite side of AB (Fig. 166)

Then the angle at P subtended by AB is LPY, the supplement of LPM, and the points P in the two cases are on the same circumference.

The complete investigation requires consideration of a *movement* of P so that P traverses this circle twice. We have considered P, moving in a clockwise direction, to pass through B. Let it continue its path so as to pass through A.

The angle at P subtended by AB (Fig. 167) is now $\angle$ XPY,

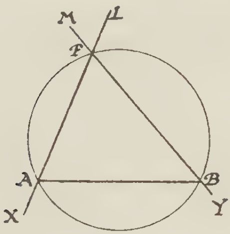


FIG. 167

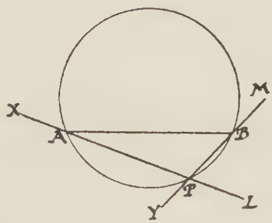


FIG. 168

and this angle = $\angle$ MPL, and therefore P is on the same arc as at first.

Let P continue its path so as to pass again through B.

The angle at P subtended by AB (Fig. 168) is now MPX which = the supplement of LPM, and P is again on the conjugate arc.

If P continues its path through A again, the cycle is repeated.

This behaviour can be illustrated by the use of an X-shaped

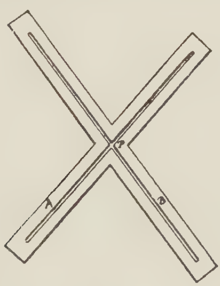


FIG. 169

frame. Slots along the arms represent the lines MPY, LPX. Pegs in a plane board represent A and B. The frame can be laid on the board so that the pegs are in the slots and moved.

The point of intersection of the slots, corresponding to P, will traverse a circle.

Now consider a $\triangle$ AQB such that AB is fixed and $\angle$ AQB is given in magnitude, the locus of Q is (1) for first considerations, an

arc ACQB, (2) for more general considerations, the circumference of the circle ACQB.

(1) To find the locus of P where P is a point in AQ produced so that QP = QB.

It is an arc AOPB such that ∠ APB = ½ ∠ AQB.

In the particular case where Q coincides with B, P also coincides with B.

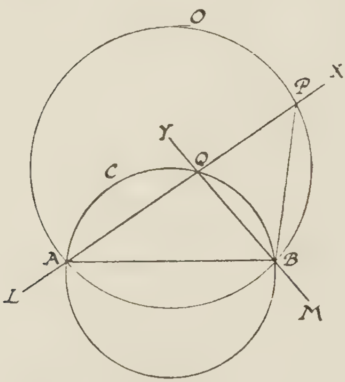


Fig. 170

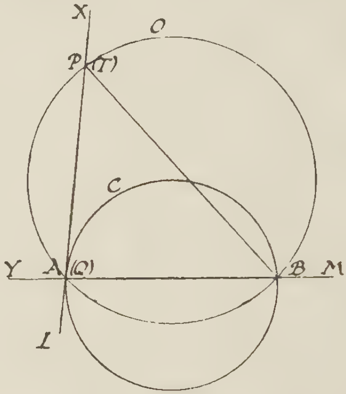


Fig. 171

(2) In the case where Q coincides with A (Fig. 171) AQP is a tangent to the circle ACB, QB coincides with AB, and AP = AB, ∵ QP = QB.

∠ APB = ∠ ABP = ½ ∠ LAB = ½ ∠ ACB = ½ the given value of AQB.

Therefore as Q moves round the arc ACB from B to A, P moves round only part of the arc BOA, viz. BOT; where T is the position of P when AP is tangential to ACB.

How are we to account for the arc TA?

We must re-state the proposition; instead of considering a △AQB with a constant angle AQB, we will take Q as the point of intersection of two straight lines LQX and MQY, which always pass through A and B respectively and which are inclined at a constant angle.

(3) Now let AQB be a position such that LQX cuts the arc TA (Fig. 172), Q being on the arc ADB conjugate to ACB; and let us *define* P *as obtained not by producing* AQ *but by cutting off from* QX *a part* QP *equal to* QB.

Then ∠ APB = ∠ QBP
= ½ ∠ PQY
= ½ suppt. ∠ XQM
= ½ ∠ ACB;

∴ P is on the arc AOB.

Now let Q, moving round the circumference of the circle BCA reach the mid-point of the arc conjugate to the arc ACB (Fig. 173).

Then QA = QB; ∴ P coincides with A. And the arc TA is accounted for.

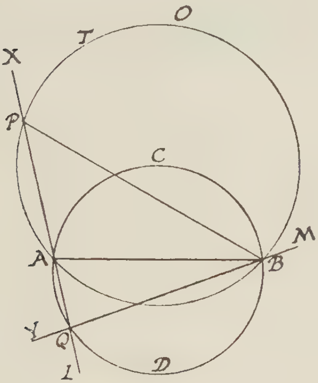


FIG. 172

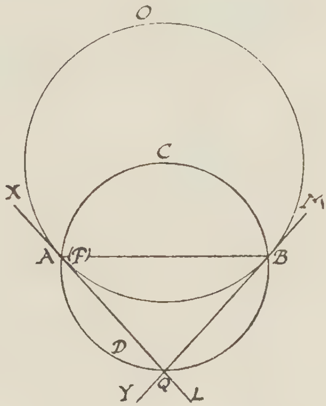


FIG. 173

[The last position of P is confirmed by the following argument:

$$\angle \text{QAB} = \tfrac{1}{2} \text{ suppt. } \angle \text{AQB}$$
$$= \tfrac{1}{2} \angle \text{ACB}$$
$$= \angle \text{AOB};$$

∴ XAQ is a tangent to the circle BOA;

∴ P must coincide with A.]

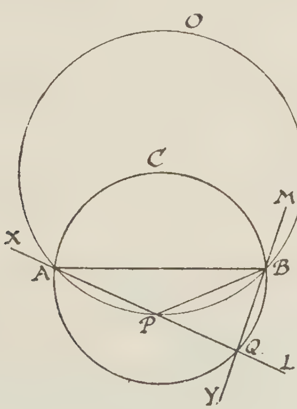


FIG. 174

(5) Let Q move round the conjugate arc farther towards B.

Then suppt. $\angle$ APB $= \angle$ BPQ
$= \frac{1}{2}$ suppt. $\angle$ PQB
$= \frac{1}{2} \angle$ ACB
$= \angle$ AOB,

and P is on the arc conjugate to BOA.

(6) When Q coincides with B, P also coincides with B.

Thus, when Q has traversed its locus circle once, P has traversed its locus circle once. When Q makes its second circuit, P makes a second circuit, X and Y occupying the former positions of L and M and vice versa.

We may put briefly the behaviour of Q and P thus: they lie on a line through A which rotates about A so that they move along the circumference of two circles which intersect in A and B.

We have thus accounted for complete and continuous loci for P and Q; but to do so, we have had to re-state the original problem.

In-centre and Ex-centre Loci.—The reader is advised to consider from this point of view the locus of the in-centre P of the $\triangle$ AQB, where AB is fixed and $\angle$ AQB is given.

With this statement of the proposition the locus is only an arc of a circle.

But it can be stated thus: Let XAL and YBM be two straight lines, passing respectively through two fixed points A and B and intersecting at Q, so that $\angle$ LQM is constant (as in Fig. 170). It is required to find the locus of P, the point of intersection of bisectors of the angles XAB and YBA.

The locus of P is a complete circle traced continuously as Q traces out its circle, but Q must traverse its locus twice. To show this and to connect the four parts of P's locus with the corresponding parts of Q's locus is left as an exercise to the reader.

But not only does the new form of statement introduce continuity, it connects the in-centre (as defined above) with an ex-centre, defined as the intersection of the bisectors of the angles LAB and MBA. The two points have the same locus circle.

The reader is also advised to consider the locus of the **ortho-centre** of the $\triangle$ AQB, where A and B are fixed and the $\angle$ AQB is constant.

There are, however, other ways of dealing with these loci. The application of some derived property of the point under consideration will give the problem a concrete bearing.

Thus, for the ortho-centre:—

Let AQB (Fig. 175) be the triangle as before, P the ortho-centre.

It can be proved that if S is the circum-centre and SE perpendicular to AB, then QP = 2SE and is of constant length, and it is perpendicular to AB.

That is, if we consider the circle ACB as being in a vertical plane and QP a rod of given length hanging vertically, then, as Q

traces out one circle, P traces out an equal circle having AB as a common chord, and this is the locus of the ortho-centre.

Again, let QP meet the circle AQB in R and AB in D (Fig. 176).

Then PD can be shown to be equal to RD.

That is, R is the optical reflection of P in the mirror AB (silvered on both sides), and P traverses its locus circle, preserving this relationship to R, and so the locus circle of P is symmetrical about AB with the locus circle of Q.

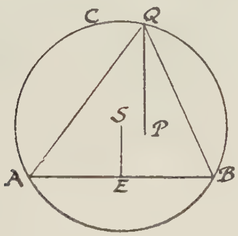


FIG. 175

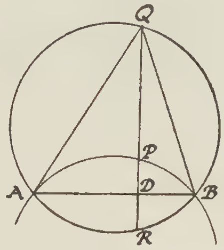


FIG. 176

For the in-centre P_1 and the ex-centre P_2, of a $\triangle$ AQB, Q, P_1, and P_2 are collinear, QP_1P_2 being the bisector of the $\angle$ AQB.

Let it meet the conjugate arc AFB in F.

Then F is the mid-point of the arc, and it may be shown that $FA = FP_1 = FB = FP_2$.

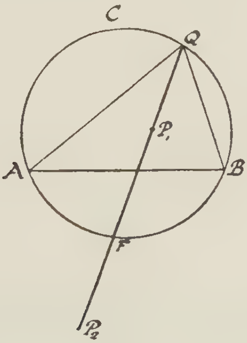


FIG. 177

If, therefore, we consider a line rotated about F so that Q describes its locus circle, P_1 and P_2 at opposite ends of a diameter sweep out simultaneously a circle of radius FA.

The reader may now consider in the same sort of way the behaviour of the other two ex-centres.

In Fig. 178, with A, Q, B, F as before, let S be the circum-centre,

I the in-centre, I_1, I_2, and I_3 the ex-centres, and let FSF_1 be a diameter of the circum-circle.

Then it is easily proved that $F_1I_2 = F_1I_3 = F_1A$. Thus while I and I_1 are extremities of a diameter of the circle through AB with F as centre, I_2 and I_3 are extremities of a diameter of the circle through AB with F_1 as centre.

A framework can be constructed to trace the three circles (the locus-circles of Q, of I and I_1, and of I_2 and I_3) simultaneously.

Rods II_1 and I_2I_3 pivoted at their mid-points F and F_1 must be arranged so that as they rotate in the plane of the figure they are at right angles at Q. (See the mechanical construction for the cissoid, p. 80.) Then Q and the points I and I_1, I_2 and I_3 trace out the locus circles.

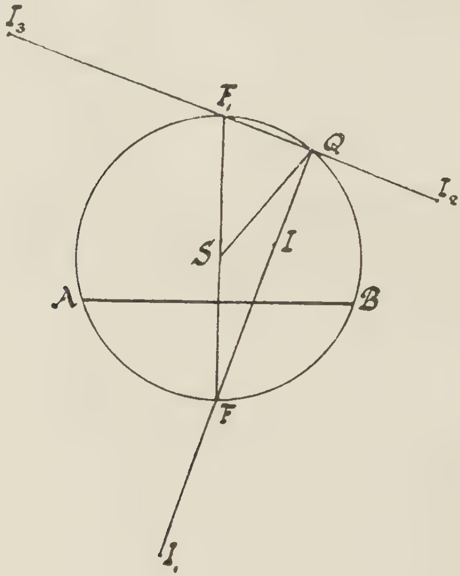


FIG. 178

In these cases we have seen that a re-statement of a proposition reduces to completeness and continuity what at first appeared as fragmentary or interrupted.

a^x.—We shall find the same in Algebra. The definition of a^m, viz., "the product of m factors each equal to a," is a definition in which m must be taken as an integer. For other values of m the definition is meaningless. But the definition leads to a rule that $a^m \times a^n = a^{m+n}$. Assuming that we take this rule as true for fractional and negative values, we can give a meaning to $x^{\frac{1}{2}}$, x^{-3}.

The mathematical instinct to apply to fractions and negative numbers the propositions that are true for integers, i.e. the endeavour to deal with continuous number, resulted in these generalizations. In the same way, the Binomial Theorem having been proved for a positive integer, Newton felt that in the absence of reason to the contrary it should be true for fractional and negative integers, and by interpolation and test convinced himself of its general truth.

With the elementary definition of a^m, the graph of a^x would be a set of points.

Let a be positive and greater than unity, and let x be restricted to positive integral values, then the graph is the set of points B, C, D, . . ., all in the first quadrant, and so placed that the eye readily "sees" the smooth curve that would pass through them and its form, if continued.

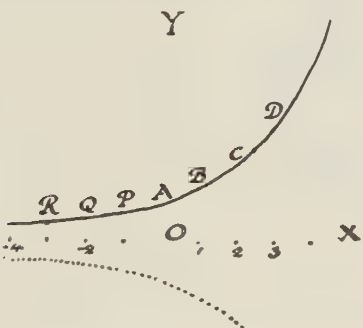


FIG. 179

To continue this smooth curve into the 2nd quadrant, x must be negative. For negative integral values, assuming the algebraical interpretation of a^{-n}, we have the points P, Q, R, . . . As these lie on the same smooth curve as B, C, D, . . ., the graph confirms the algebraical interpretation of negative indices.

This smooth curve will cut OY in A, where OA = 1, confirming that $a^0 = 1$.

Now if fractional values are given to x, again assuming the algebraical interpretation of $a^{p/q}$, we obtain other points which will be seen to lie on the same curve. By taking a sufficient number of fractional values we can obtain points as near together as we like, but the graph does not become a continuous curve unless x has all real values, including surds, logs, etc. (*see* p. 184). For all real values of x the graph of the function a^x includes the smooth, continuous curve RQPABCD.

But this is not the whole of the graph.

If $x = \frac{1}{2}$, $a^x = \pm\sqrt{a}$, i.e. there are two points corresponding to $x = \frac{1}{2}$:—one in the 1st quadrant already considered, the other

in the 4th, its optical image in the mirror X_1OX. By giving x values of the form $\frac{p}{2q}$, where p and q are integers, we can obtain as many as we like of these points which have reflections in X_1OX.

But the reflection of the curve RAD is not continuous. For, let $x = \frac{1}{3}$, then $a^x = \sqrt[3]{a}$ and has only one real value, and it is in the 1st quadrant; there is no reflection in the 4th. And whenever x is of the form $\frac{p}{2q+1}$, p and q being integers, there will be a break in the reflection of RAD.

To consider the graph further we must be able to decide on the number and nature of the values of a^x for any value of x. Here we will content ourselves with saying that the graph of a^x consists of (1) a continuous curve, (2) a reflection of it, not continuous but consisting of points separated by gaps smaller than any assignable magnitude.

Log x.—If $a^x = y$, $\log_a y = x$.

If, then, we interchange x and y in Fig. 179, we have the graph of logarithms to base a; and we see that while there is only one logarithm for any given number, there may be, but are not bound to be, two numbers (of the same magnitude but of opposite sign) for certain logarithms.

CHAPTER XIII

NEGATIVE MAGNITUDES

It is generally recognized that negative magnitudes correspond to positive magnitudes, measureable by the same number of the same units, but having an opposite direction. Thus if l represents a number of miles measured north from a given position A, $-l$ represents the same number measured south from A; if a journey l takes a traveller to B, then a further journey $-l$ takes him back to A, and $l + (-l) = 0$.

This interpretation is inherent in graphs and algebraic geometry generally; it reduces certain pairs of propositions to single ones,

Fig. 180

i.e. it makes for generalization. Thus, if AB, a straight line of a units, is produced to C, so that BC is of b units, then a line AC is constructed of $a + b$ units, and the identity $(a + b)^2 \equiv a^2 + b^2 + 2ab$ can be illustrated by the dissection of a square drawn on AC. In the geometrical enunciation we say that AC is divided internally at B.

But if from AB a part BC of b units is cut off, the line AC is constructed of $a - b$ units.

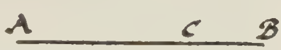


Fig. 181

AC is now said to be divided externally in B, and the identity $(a - b)^2 \equiv a^2 + b^2 - 2ab$ can be illustrated by what we might call external dissection. And the two propositions can be regarded as one if we use a notation in which a magnitude is represented in such a way that direction is involved. Thus, if $\overline{AB}$ represents a displacement in the direction from A to B, and $\overline{AB} = -\overline{BA}$, then for all points in a straight line $\overline{AB} + \overline{BC} = \overline{AC}$ and $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2\overline{AB} \,.\, \overline{BC}$.

It will in this case be necessary to interpret negative area, so as to satisfy

$$\{(a + (-b)\}^2 = a^2 + b^2 + 2a(-b) = a^2 + b^2 - 2ab,$$

where $a + (-b)$, the addition of a negative magnitude, gives the same result as $a - b$, the subtraction of a positive magnitude.

Consider ABC a triangle with a median AO, and take on AO any point D (1) within the triangle, (2) in AO produced, (3) in OA produced. That is, in (1) AO is divided internally, in (2) and (3) externally, in D.

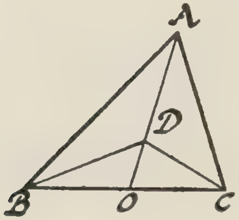


Fig. 182 (1)

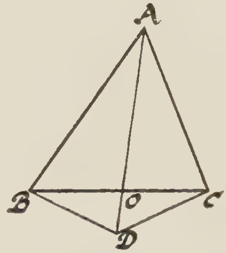


Fig. 182 (2)

Fig. 182 (3)

It is to be proved that $\triangle ABD = \triangle ADC$.

In each case, $\triangle ABO = \triangle AOC$,

$\triangle DBO = \triangle DOC$.

In (1),
$$\begin{aligned} \triangle ABD &= \triangle ABO - \triangle DBO \\ &= \triangle AOC - \triangle CDO \\ &= \triangle ADC. \end{aligned}$$

In (2),
$$\begin{aligned} \triangle ABD &= \triangle ABO + \triangle OBD \\ &= \triangle CAO + \triangle ODC \\ &= \triangle ADC. \end{aligned}$$

In (3),
$$\begin{aligned} \triangle ABD &= \triangle DBO - \triangle ABO \\ &= \triangle CDO - \triangle CAO \\ &= \triangle ADC. \end{aligned}$$

Now suppose that for these triangular areas we take as a positive magnitude one for which the letters are written in an anti-clockwise direction, we shall then, agreeably with our convention for negative magnitudes, regard as negative one for which the letters are written in a clockwise direction, and we shall have

$$\begin{aligned} \triangle \overline{ABC} &= \triangle \overline{BCA} = \triangle \overline{CAB} \\ &= -\triangle \overline{ACB} = -\triangle \overline{CBA} = -\triangle \overline{BAC}. \end{aligned}$$

Take (1) as the standard case.

Then, remembering our hypothesis of signs, we can write (2)

$$\begin{aligned} \triangle \overline{ABD} &= \triangle \overline{ABO} - \triangle \overline{DBO} \\ &= \triangle \overline{AOC} - \triangle \overline{CDO} \\ &= \triangle \overline{ADC}; \end{aligned}$$

and we can write (3)

$$\begin{aligned} -\triangle\,\overline{ABD} &= \triangle\,\overline{DBO} - \triangle\,\overline{ABO} \\ &= \triangle\,\overline{CDO} - \triangle\,\overline{AOC} \\ &= -\triangle\,\overline{ADC}\,; \end{aligned}$$

that is, all three cases could be included in the form of proof :

$$\begin{aligned} \triangle\,\overline{ABD} &= \triangle\,\overline{ABO} - \triangle\,\overline{DBO} \\ &= \triangle\,\overline{AOC} - \triangle\,\overline{CDO} \\ &= \triangle\,\overline{ADC}. \end{aligned}$$

This extends the convention of signs for points along a straight line to three points arranged cyclically. It applies to angles as to triangles. Thus in the proposition that the angle at the centre of a circle is double of the angle at the circumference on the same arc, we can dispense with separate cases, providing that we keep to the convention of signs and write in both figures

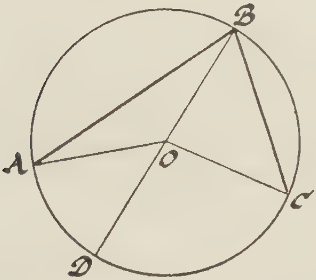


FIG. 183 (1)

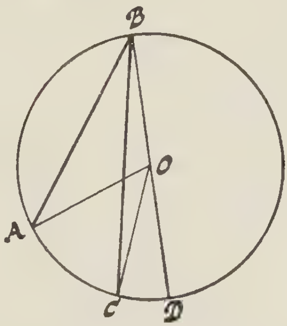


FIG. 183 (2)

$$\begin{aligned} \angle\,\overline{COD} &= 2\,\angle\,\overline{CBO}, \\ \angle\,\overline{DOA} &= 2\,\angle\,\overline{DBA}, \\ \angle\,\overline{COD} + \angle\,\overline{DOA} &= 2\,\angle\,\overline{CBO} + 2\,\angle\,\overline{DBA}, \\ \angle\,\overline{COA} &= 2\,\angle\,\overline{CBA}. \end{aligned}$$

We have thus been led to consider negative angles, and we shall return to this later.

In the case of the areas of triangles, we could also have used the convention that if $\triangle$ BDC on one side of BC is regarded as positive, it may be regarded as negative if on the other side ; $\triangle$ ABD regarded as positive on one side of AB is to be regarded as negative on the other side, i.e. we could regard the sign of the area as changed if the triangle is rotated about one side through an angle of 180°. This convention, too, is of wide and consistent application ; it appears, e.g., in the estimate of areas of triangles in algebraic geometry, when they are found in terms of the co-ordinates of their vertices. We will now apply it to the case of rectangles.

Take XOX_1 and YOY_1 as axes of co-ordinates, P a point and PMON a rectangle.

Consider PMON in the 1st quadrant as having positive area. After rotation about YO into the 2nd quadrant it should be negative. Its area in the 1st quadrant is xy, x and y both positive; in the second x is negative and y positive, so that the product is negative.

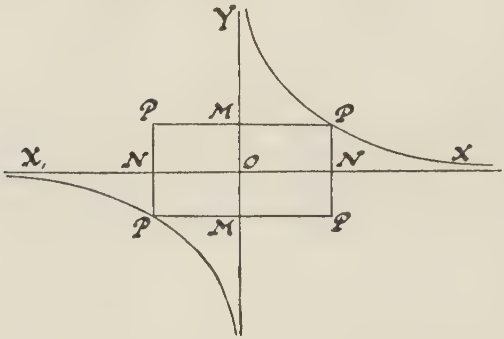


FIG. 184

Rotation from the 2nd to the 3rd quadrant about XOX_1 should give another change of sign, and the area should be positive; in this case both x and y are negative, and the product, being positive, satisfies the requirements of the convention, and so on.

These results are all consistent with the statements—

(1) The area of PMON is $\overline{OM} \cdot \overline{ON}$.

(2) The locus of P for all positions such that the area PMON has a positive area, say c^2, is the rectangular hyperbola $xy = c^2$ with its two branches in the 1st and 3rd quadrants, and the complementary locus for a negative area, say $-c^2$, is the conjugate

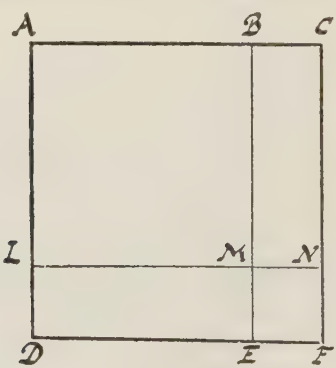


FIG. 185 (1)
$(a + b)^2$

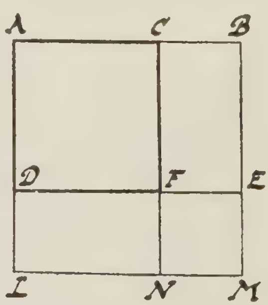
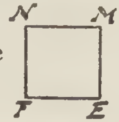
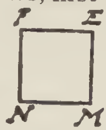


FIG. 185 (2)
$(a - b)^2$

hyperbola $xy = -c^2$, having branches in the 2nd and 4th quadrants.

In the case of the $(a+b)^2$ and $(a-b)^2$ formulæ draw the corresponding figures. See Figs. 185 (1), (2).

For $(a+b)^2$, Fig. 185 (1), AC^2 is AF = AM + MF + BN + LE; to obtain from it $(a-b)^2$, AF and AM remain unchanged in position; MF must be twice rotated thus, first about ME to give 

and then about MN to give 

, BN must be once rotated (about BM), and LE must be once rotated (about LM), giving AF = AM + MF − BN − LE (all the magnitudes being positive).

It will also be noticed that the rotational convention of sign applied to the triangle (p. 196) may equally be applied here. Thus in both formulæ ADFC and MEFN are anti-clockwise, and therefore of the same sign in each; whereas BMNC and LDEM are anti-clockwise in the $(a+b)^2$ and clockwise in the $(a-b)^2$ formula, and will therefore change signs in passing from the one formula to the other.

Returning to the sign of angles, we shall adduce one or two corroborative cases.

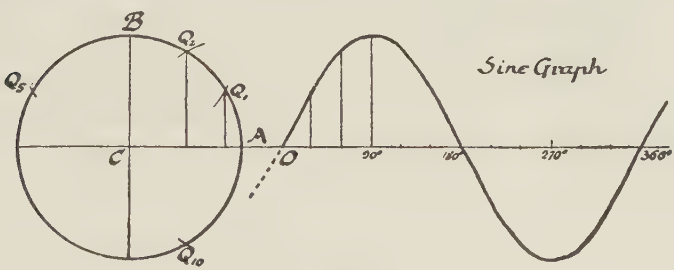


FIG. 186

Obtaining the sine graph by projection along OY of a radius rotating anti-clockwise from CA, which lies along OX, we can extend it indefinitely in the positive direction by considering the rotation to continue indefinitely, i.e. if we admit angles of any magnitude. We can also extend it backwards as the dotted line, if we consider CA to rotate clockwise, i.e. in a negative direction. The continuity of the periodic graph requires a hypothesis not only of angles of any magnitude but also of negative angles.

Assuming the formula

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

to be established for acute angles and extending it to negative angles, we have

$$\sin\{A + (-B)\} = \sin A \cos(-B) + \cos A \sin(-B),$$

and using the fact that $\sin(-B) = -\sin B$, and $\cos(-B) = \cos B$, as can be illustrated in the sine and cosine graphs, we get

$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

And this can be established independently. Let these formulæ be proved thus:

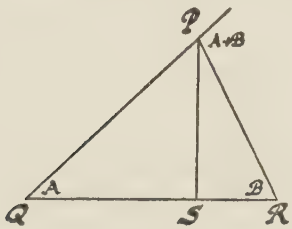


FIG. 187

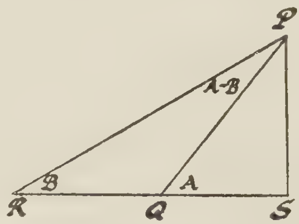


FIG. 188

Draw ∠s PQR and PRQ equal to A and B, and let QP and RP meet in P (Fig. 187). Draw PS perpendicular to QR.

Then $\qquad QS = PQ \cos A$

$\qquad\qquad RS = RP \cos B$

and $\qquad QR = PQ \cos A + RP \cos B$;

writing for QR, PQ, RP their values in terms of d (the diameter of the circum-circle PQR),

$$d \sin(A + B) = d \sin B \cos A + d \sin A \cos B,$$

i.e. $\qquad \sin(A + B) = \sin A \cos B + \cos A \sin B.$

For $\sin(A - B)$ draw the corresponding Fig. 188, in which, instead of two interior angles of a triangle, one exterior and one interior are used for A and B.

Then $\qquad RQ = RS - QS$

$$d \sin(A - B) = d \sin A \cos B - d \sin B \cos A.$$

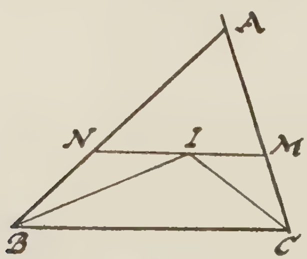


FIG. 189

This relationship of the interior and exterior angles of a triangle has already been illustrated in the case of the locus of in-centre and ex-centre (p. 190). Another example is included

here. To draw a line parallel to the base BC of a $\triangle$ ABC cutting AB and AC in N and M, so that NB + MC = NM (Fig. 189).

Let the internal bisectors of the angles at B and C meet in I; through I draw NIM parallel to BC (Fig. 189).

Then NI = NB and IM = MC and NB + MC = NM.

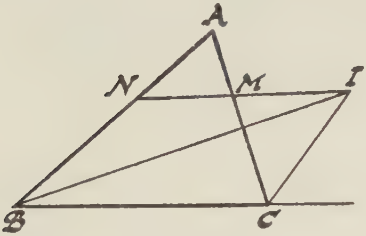


FIG. 190 (1)

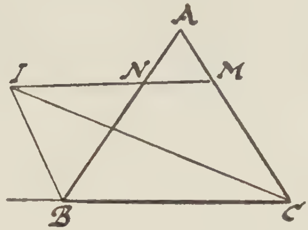


FIG. 190 (2)

If I is obtained by bisecting one exterior and one interior angle, NM = NB $\sim$ MC, and all cases are included under the general problem, to draw NM so that it is the sum or difference of NB and MC. This brings us back to another point of view of the formulæ for $(a + b)^2$, $(a - b)^2$, sin (A + B), sin (A − B), viz., the resemblance in the formulæ for functions of sum and difference. And of this another example is given:

If OX and OY are two lines at right angles, and P is such that the sum of its $\perp$s PM and PN on the lines is constant $(= c)$, the locus of P is AB, a line equally inclined to OX and OY.

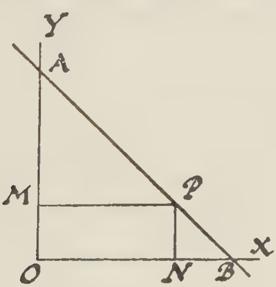


FIG. 191

But the extensions of AB in both directions form the locus of P such that PM $\sim$ PN $= c$ and the whole line is the locus of points P for which the sum or difference of PM and PN is c. Its equation is $x + y = c$, and it can be regarded as the locus of P such that PM + PN $= c$, if we adhere to the convention of sign for PM and PN.

Lastly, if a concrete problem gives a negative solution, we may take it that we have assumed for the unknown a direction opposite

to the true one. Thus two brothers are aged 16 and 12 years. When *will* the one's age be double the other's? Let it be *after* x years.

Then $$16 + x = 2(12 + x),$$
and $$x = -8.$$
We assumed the time in the future; it should be 8 years in the past, i.e. the elder *was* twice as old as the younger 8 years ago.

In the following geometrical problem let AB, BC, CA be 13, 9, 7 units long, and let AD be perpendicular to BC; it is required to find the length CD.

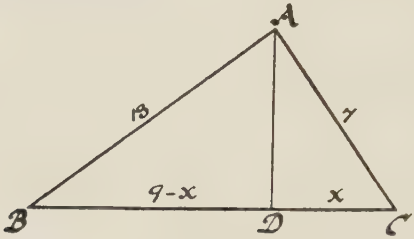


FIG. 192

Taking the figure as drawn and letting CD be x, we get
$$13^2 - (9 - x)^2 = 7^2 - x^2,$$
$$13^2 - 7^2 = (9 - x)^2 - x^2,$$
$$(13 + 7)(13 - 7) = (9 - 2x)9,$$
$$120 = (9 - 2x)9,$$
$$13\tfrac{1}{3} = 9 - 2x,$$
$$4\tfrac{1}{3} = -2x,$$
$$x = -2\tfrac{1}{6};$$
i.e. CD should be measured in the other direction, as in the Figure 193.

FIG. 193

In trigonometry, given $c = 10$, $b = 14$, $\angle B = 60°$.

Then $$b^2 = c^2 + a^2 - 2ca \cos B,$$
$$14^2 = 10^2 + a^2 - 10a,$$
$$a^2 - 10a - 96 = 0,$$
$$(a - 16)(a + 6) = 0,$$
$$a = 16 \text{ or } -6.$$

The Figs. 194 (1) and (2) show the solution; in Fig. (2) BC must be measured in the opposite direction to BC in Fig. (1), and $\angle$ B for one triangle is the exterior angle for the other. The solution gives both points, C_2 and C_1, that would be obtained in the construction for a geometrical solution.

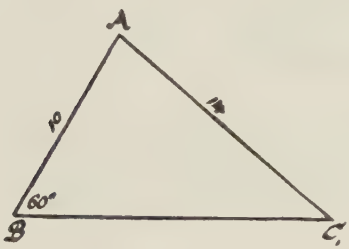


FIG. 194 (1)

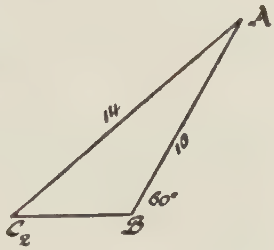


FIG. 194 (2)

In dynamics the equation

$$160 = -48t + 16t^2$$

solves the problem, *A particle is projected vertically upwards from the top of a cliff with a velocity of* 48 *feet per second; when will it reach the beach* 160 *feet below?*

$$t = 5 \text{ or } -2 \text{ seconds.}$$

The solution, 5 seconds, gives the time for the particle to reach its highest point, turn, and fall to the beach; the solution, -2 seconds, means that if two seconds before the time of projection the particle had started upwards from the beach, it would have passed the cliff-top with an upward velocity of 48 feet per second. The solution of the equation gives the times for the two parts of its flight: (1) from the beach to the cliff-top, (2) from the cliff-top to the beach again.

Thus the convention of sign not only generalizes mathematical analysis but also corrects automatically a wrong assumption of direction, and completes the interpretation of results, which otherwise would be only partially accountable.

CHAPTER XIV

COMPLEX NUMBERS

NEGATIVE numbers arise when algebra exercises its function of generalizing arithmetical processes. The solution of the equation $b + x = a$ tells us what number, supposed x, must be added to b to give a; in arithmetic b is necessarily $\not> a$, in algebra b is unlimited in magnitude, and may be greater than a, in which case x is negative.

The simple equation introduces the necessity either of disregarding negative numbers as meaningless—Stevin called them *numeri absurdi*—or of interpreting them. An interpretation has been found which can be consistently applied.

The quadratic equation introduces a new sort of number. $x^2 + 4 = 0$ is not satisfied by any positive or negative value for x; we can write the solution as $x = \pm 2\sqrt{-1}$, or (using i, a symbol due to Euler, for $\sqrt{-1}$,) as $x = \pm 2i$. No numerical value can be assigned to it, and we might be content to regard it as merely an algebraical abstraction, but we may do more than that: we may look for some means of interpreting it, or at least of seeing a significance in it.

Let it be required to find the third side of a triangle given $c = 10$, $b = 7$, $B = 60°$.

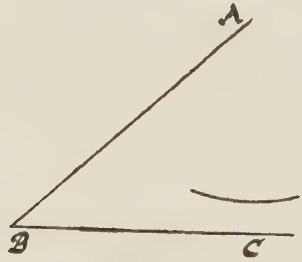


FIG. 195

$$b^2 = c^2 + a^2 - 2ca \cos 60°,$$
$$49 = 100 + a^2 - 10a,$$
$$a^2 - 10a + 51 = 0,$$
$$a = 5 \pm \sqrt{-26}, \text{ or}$$
$$5 \pm \sqrt{26} . i.$$

If the geometrical solution is attempted it is found to fail,

that is to say, the circle with A as centre and 7 as radius does not cut BC in real points.

This is in general the significance of the appearance of i in the solution of a geometrical problem: that two lines whose points of intersection should give the solution do not intersect in real points. We use for convenience the phrase, "they intersect in imaginary points." These imaginary points occur in pairs.

The intersection of a straight line and circle affords a simple illustration.

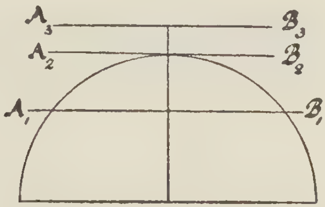


Fig. 196

If $\qquad x^2 + y^2 = a^2$ is the equation of the circle
and $\qquad y = d$ the equation of the line,

$$x^2 = a^2 - d^2,$$
$$x = \pm\sqrt{a^2 - d^2}.$$

If $d < a$, as for A_1B_1, the roots are real and can be found arithmetically.

If $d = a$, as for A_2B_2, $x = \pm 0$.

If $d > a$, as for A_3B_3, $x = \pm\sqrt{d^2 - a^2} \,.\, i$.

We say that A_1B_1 cuts the circle in two real points, A_2B_2 in two coincident points, and A_3B_3 in two imaginary points.

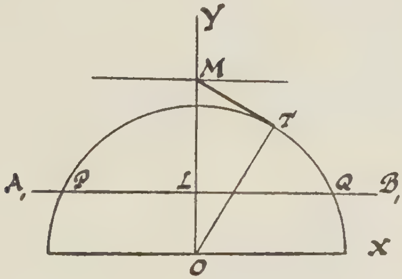


Fig. 197

Now if OY meets A_1B_1 in L, and A_1B_1 meets the circumference in P and Q, PQ is a chord distant d from the centre, and x the semi-chord $= \sqrt{a^2 - d^2}$.

If OY meets A_3B_3 in M, and MT is a tangent, then MT $= \sqrt{d^2 - a^2}$ and $x = \sqrt{d^2 - a^2} \,.\, i =$ tangent $\times\, i$.

In the case of the chord PQ, the semi-chord is real and the tangent

$$= \sqrt{d^2 - a^2} \text{ (by general formula).}$$
$$= \sqrt{a^2 - d^2}\,.\,i = \text{semi-chord} \times i,$$

i.e. for a point inside the circle there is a real semi-chord but an imaginary tangent, for a point outside the circle the semi-chord becomes imaginary as the tangent becomes real; and wherever the point is, the line which is imaginary is i times the line which is real.

The same connexion between semi-chord and tangent appears in the proposition that if any line through a fixed point X meet a fixed circle in A and B, the rectangle AX . XB is constant.

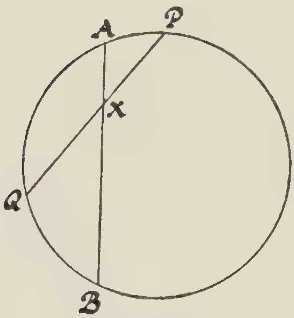


FIG. 198 (1)

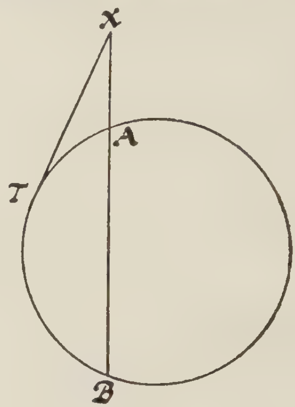


FIG. 198 (2)

Where X is within the circle, let PXQ be the chord which is bisected at X; where X is outside the circle let XT be the tangent.

Then $\qquad XP^2 = AX\,.\,XB = -\,XB\,.\,XA,$

$\qquad\qquad XT^2 = XB\,.\,XA = -\,AX\,.\,XB;$

the expression for $XP^2 = -$ the expression for XT^2, and
the expression for $XT^2 = -$ the expression for XP^2.

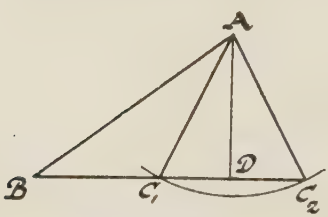


FIG. 199 (1)

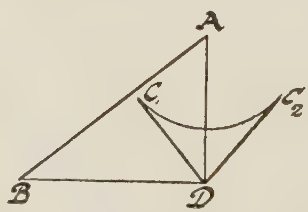


FIG. 199 (2)

This connexion is applicable to the problem we considered on page 204.

To construct a triangle given c, b, and B, we draw an $\angle$ ABD of the magnitude B, make BA of length c, and with A as centre and radius b, describe a circle. If the circle cuts BD in C_1 and C_2, the triangle satisfying the data is ABC_1 or ABC_2. [Fig. 199 (1).]

Now let D be such that AD is perpendicular to BD.

Then $a = BD - C_1D$ or $BD + DC_2$.

But if the circle does not meet BD, from D (where AD is perpendicular to BD) draw tangents DC_1 and DC_2 to the circle. [Fig. 199 (2).]

Then for the data $c = 10$, $b = 7$, $B = 60°$, $BD = 5$, $DC_1 = DC_2 = \sqrt{26}$ and $a = BD - i \,.\, C_1D$ or $BD + i \,.\, DC_2$.

Thus we see that not only is there a practical significance in the appearance of i in the solution of a geometrical problem but that such a solution may in some cases have a precise geometrical interpretation.

We may inquire further whether the manipulation of i in accordance with the laws of algebra leads to results consistent with general formulæ.

Taking the same data $c = 10$, $b = 7$, $B = 60°$, the sine formula gives, for C,

$$\sin C = \frac{c}{b} \sin 60° = \frac{5\sqrt{3}}{7}$$
$$= 1{\cdot}237 \ldots$$

Now the sine graph lies between the limits $y = 1$ and $y = -1$, so the line $y = 1{\cdot}237$ does not cut it in real points, i.e. there is no real angle whose sine satisfies the conditions. But though the triangle cannot be constructed an expression has been found for the third side and for the sine of the second angle, and it may be noticed incidentally that though this angle is imaginary it has a real sine.

a has already been found.

Of the two imaginary triangles corresponding to the two values of a, consider the one in which $a = 5 + \sqrt{26} \,.\, i$.

$$\text{For A, } \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{15 - \sqrt{26} \,.\, i}{14},$$

$$\sin A = \frac{a \sin B}{b} = \frac{\sqrt{3}}{14}(5 + \sqrt{26} \,.\, i).$$

$$\text{For C, } \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{-1 + 10\sqrt{26} \,.\, i - 51}{14(5 + \sqrt{26} \,.\, i)}$$
$$= \frac{-26 + 5\sqrt{26} \,.\, i}{7(5 + \sqrt{26} \,.\, i)}$$
$$= \frac{\sqrt{26} \,.\, i\,(5 + \sqrt{26} \,.\, i)}{7(5 + \sqrt{26} \,.\, i)}$$
$$= \frac{\sqrt{26} \,.\, i}{7}.$$

Using these values we get

$$\sin^2 A + \cos^2 A = \frac{3}{196}(-1 + 10\sqrt{26}\,.\,i) + \frac{199 - 30\sqrt{26}\,.\,i}{196}$$
$$= 1.$$
$$\sin^2 C + \cos^2 C = \frac{75}{49} - \frac{26}{49}$$
$$= 1.$$
$$\sin(A + C) = \sin A \cos C + \cos A \sin C$$
$$= \frac{\sqrt{3}(5 + \sqrt{26}\,.\,i)}{14} \cdot \frac{\sqrt{26}\,.\,i}{7} + \frac{15 - \sqrt{26}\,.\,i}{14} \cdot \frac{5\sqrt{3}}{7}$$
$$= \frac{-26\sqrt{3} + 75\sqrt{3}}{14 \times 7}$$
$$= \frac{\sqrt{3}}{2}$$
$$= \sin B$$
$$= \sin(180^\circ - \overline{A + C}).$$
$$\cos(A + C) = \cos A \cos C - \sin A \sin C$$
$$= \left(\frac{15 - \sqrt{26}\,.\,i}{14}\right)\frac{\sqrt{26}\,.\,i}{7} - \frac{\sqrt{3}}{14}(5 + \sqrt{26}\,.\,i)\frac{(5\sqrt{3})}{7}$$
$$= \frac{26 - 75}{7\,.\,14}$$
$$= -\tfrac{1}{2}$$
$$= -\cos B.$$
$$= -\cos(180^\circ - \overline{A + C}).$$

That is to say, the identities which hold for real angles hold also for imaginary angles.

The reader is advised to work out from the cosine formula and Heron's formula, and from the formulæ for ratios of $\frac{A}{2}$, etc., expressions for the angles of the imaginary triangle whose sides are 3, 8, 13, and to test any identities he knows for the values obtained. He will confirm the conclusion at which we have just arrived.

Representation—The Argand Diagram.—All these expressions involving i are of the form $a + b\,.\,i$. They are called complex numbers. Let us now see if we can find some way of representing them.

Consider a line of unit length OA; rotate it through 180° to the position OA_1. OA_1 represents -1.

Now consider that the rotation is the result of two equal rotations—one through 90° to OB, and the other through 90° to OA_1. Let us further suppose that the rotation through 180° represents the result of multiplying by -1, and that rotation

through 90° represents the result of multiplying by x. Then OB represents OA . x and OA_1 represents OB . x, i.e. OA . x^2.

Then $$-1 = 1 . x^2 \text{ and } x = i \text{ or } -i.$$

We will take i to represent an anti-clockwise rotation and $-i$ to represent a clockwise rotation through 90°; then $OB_1 = OA \times (-i)$, i.e. it represents $-i$. Again, OB_1 is $OA_1 \times i$, i.e. it represents $-i$. These results are consistent.

The solution of $x^2 = 1$ is $x = 1$ or -1, and is represented by OA and OA_1, i.e. the two radii which divide a complete circular rotation into 2 equal parts represent the solution of an equation of the 2nd degree or the 2 square-roots of unity.

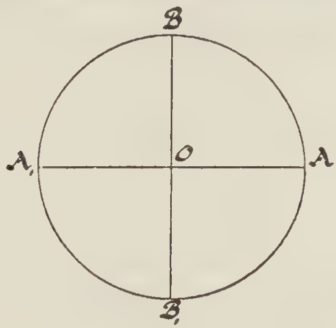


FIG. 200

The solution of $x^4 = 1$ is $x^2 = 1$ or -1, i.e. $x = 1, -1, i, -i$, and this is represented by OA, OB, OA_1, OB_1. So that the 4 radii which divide a complete circular rotation into 4 equal parts represent the solution of a 4th degree equation or the 4 fourth-roots of unity.

Now consider the equation $x^3 = 1$, which should give the 3 cube-roots of unity.

$$x^3 - 1 = 0 \text{ gives } (x - 1)(x^2 + x + 1) = 0,$$

and the solutions are

$$x = 1 \text{ or } \frac{-1 \pm \sqrt{3}.i}{2}.$$

If one imaginary root is called ω, it does not matter which, the other will be found by multiplication to be ω^2. This is not an accident, for, as we shall see presently, if α is one imaginary root of $x^n = 1$, the others are of the form α^r, where r is an integer, and one of these can be chosen and called ω so that the others are $\omega^2, \omega^3, \omega^4, \ldots \omega^{n-1}$.

The roots ω and ω^2 cannot be evaluated; they have no place in our number scale, for they $\neq 1$ and, if they are real numbers either $>$ or $<$ 1, then x^3 is correspondingly much $>$ or $<$ 1. The question is whether they can be represented in the same sort of way as the square-roots and the fourth-roots of unity. If so, the

three radii which divide the complete circular rotation into 3 equal parts will represent 1, ω, and ω^2. For if multiplication by ω is represented by rotation through 120°, then rotation through 240° represents multiplication by ω^2, and rotation through 360° represents multiplication by ω^3, or 1.

Take OC and OD so that $\angle AOC = \angle COD = \angle DOA = 120°$.

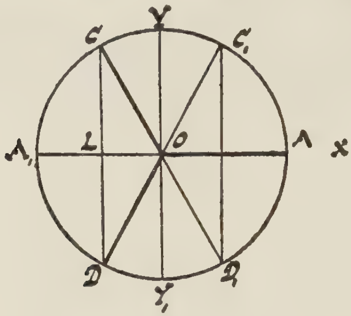


FIG. 201

CD is perpendicular to AA_1 at L, and $OL = \frac{1}{2}$, and $CL = \frac{\sqrt{3}}{2}$.

OL in the negative direction represents $-\frac{1}{2}$

LC in the i direction represents $\frac{\sqrt{3} \cdot i}{2}$,

i.e. the displacement $\overline{OC}$, made up of the displacements $\overline{OL}$ and $\overline{LC}$, represents $\frac{-1 + \sqrt{3} \cdot i}{2}$, and this is ω. Similarly, OD represents $\frac{-1 - \sqrt{3} \cdot i}{2}$, and this is ω^2. And $\overline{OA}$ is 1, which is ω^3.

In the same way it is easy to show that the roots of $x^3 = -1$ are represented by OA_1, OC_1, OD_1 where $\angle C_1OA_1 = \angle A_1OD_1 = \angle D_1OC_1$; and the roots of $x^6 = 1$ are represented by OA, OC_1, OC, OA_1, OD, OD_1.

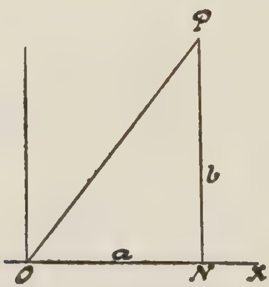


FIG. 202

This mode of representation applied to $a + b \,.\, i$ would add a displacement NP of b in the i direction to a displacement a in the positive direction ON, i.e. OP in the figure would represent $a + b \,.\, i$.

This representation (Fig. 202) due to Argand (A.D. 1806) is called an "Argand Diagram." The representation of i by OY had been devised by Caspar Wessel (1797).

Now OP is $\sqrt{a^2 + b^2}$ in length, and $a + b \,.\, i$ can be written $\sqrt{a^2 + b^2}\,(\cos\alpha + i \,.\, \sin\alpha)$, where α is $\angle$ POX.

That is to say, when a line of length OP has been rotated through an angle α it represents in its new position OP $(\cos\alpha + i\sin\alpha)$, and rotation through an angle α represents multiplication by $\cos\alpha + i\sin\alpha$.

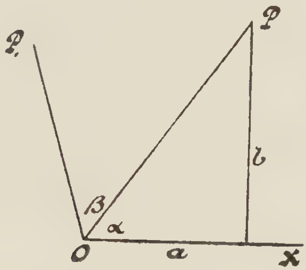


FIG. 203

If the Argand representation is to give consistent results, successive rotations through angles α and β must produce the same expression as rotation through one angle $\alpha + \beta$.

Draw OP_1 of length OP and such that $\angle POP_1$ is β (Fig 203).

Then OP represents $\sqrt{a^2 + b^2}\,(\cos\alpha + i\sin\alpha)$,
OP_1 represents OP $(\cos\beta + i\sin\beta)$,
i.e. $\sqrt{a^2 + b^2}\,(\cos\alpha + i\sin\alpha)\,(\cos\beta + i\sin\beta)$,
i.e. $\sqrt{a^2 + b^2}\,\{\cos\alpha\cos\beta - \sin\alpha\sin\beta + i\,(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\}$,
i.e. $\sqrt{a^2 + b^2}\,\{\cos(\alpha + \beta) + i\sin(\alpha + \beta)\}$,
or the result of rotating $\sqrt{a^2 + b^2}$ through an angle $\alpha + \beta$.

This is the foundation of **De Moivre's Theorem.**

In particular if $\beta = \alpha$,

$$\cos 2\alpha + i\sin 2\alpha = (\cos\alpha + i\sin\alpha)^2,$$

and so
$$\cos 3\alpha + i\sin 3\alpha = (\cos\alpha + i\sin\alpha)^3,$$

and finally
$$\cos n\alpha + i\sin n\alpha = (\cos\alpha + i\sin\alpha)^n.$$

If $n\alpha = 360°$, i.e. if a complete rotation is divided into n parts each equal to α, n being a positive integer,

$$\cos n\alpha + i\sin n\alpha = \cos 360° + i\sin 360°$$
$$= 1.$$

Therefore
$$1 = (\cos\alpha + i\sin\alpha)^n;$$

and $\cos\alpha + i\sin\alpha$ is an nth root of unity, $\cos 2\alpha + i\sin 2\alpha$ is another, for

$$\begin{aligned}(\cos 2\alpha + i\sin 2\alpha)^n &= \{(\cos\alpha + i\sin\alpha)^2\}^n,\\ &= \{(\cos\alpha + i\sin\alpha)^n\}^2\\ &= 1;\end{aligned}$$

i.e. the nth roots of unity are $(\cos\alpha + i\sin\alpha)$, $(\cos\alpha + i\sin\alpha)^2$, etc., where $\alpha = \frac{360°}{n}$. And this shows that the construction of an angle of $\frac{360°}{n}$ is connected with the solution of $x^n = 1$, and so with the factorization of $x^n - 1 = 0$.

De Moivre's Theorem may be extended to cases when n is not a positive integer; but we will not pursue that here.

It may now be asked, "Are there other expressions arising out of algebraical equations to confront the mathematician? Negative and complex numbers have in some way been brought satisfactorily within the fold of the laws of algebra; do others remain outside, perhaps waiting for inclusion? What of $\sqrt{a + ib}$, of $\log(a + ib)$, etc?" The answer is that they can all be reduced to the form $a + ib$. There is nothing more difficult or complicated than this form.

Let us consider some cases.

EXAMPLE 1.—To find an expression for $\sqrt{i}$.

If $x = \sqrt{i}$, then $x^2 = i$, $x^4 = -1$, $x^8 = 1$; and one value of $\sqrt{i}$, to judge by the Argand representation, is $\cos 45° + i\sin 45°$, i.e. $\frac{\sqrt{2} + \sqrt{-2}}{2}$.

This is of the form $a + ib$.

It could be obtained algebraically thus

$$\begin{aligned}&x^8 - 1 = 0 \quad \text{gives } x^4 + 1 = 0 \text{ or } x^4 - 1 = 0;\\ &x^4 - 1 = 0 \quad \text{gives } 1, -1, i, -i, \text{ already dealt with,}\\ &x^4 + 1 = 0 \quad \text{gives } \sqrt{i}, -\sqrt{i}, \sqrt{-i}, -\sqrt{-i}.\end{aligned}$$

But to solve $x^4 + 1 = 0$, we may write it

$$x^4 + 2x^2 + 1 - 2x^2 = 0,$$

i.e. $$(x^2 + 1)^2 - (x\sqrt{2})^2 = 0,$$

i.e. $$x^2 + x\sqrt{2} + 1 = 0,$$

or $$x^2 - x\sqrt{2} + 1 = 0,$$

of which the solutions are

$$x = \frac{-\sqrt{2} \pm \sqrt{-2}}{2} \text{ and } x = \frac{\sqrt{2} \pm \sqrt{-2}}{2}.$$

These four roots are, then, values of $\pm\sqrt{\pm i}$, expressed in the form $a + ib$. In this form the reader can identify them with the values represented by 4 of the 8 radii which divide a complete circular rotation into 8 equal parts.

EXAMPLE 2.—To express $\sqrt{p + iq}$ in the form $a + ib$.

For $\sqrt{p^2 + q^2}$ write r.

Then
$$p + iq = r(\cos \alpha + i \sin \alpha),$$
where $\cos \alpha$ is $\dfrac{p}{\sqrt{p^2 + q^2}}$,
$$\sqrt{p + iq} = \pm \sqrt{r}\left(\cos \frac{\alpha}{2} + i \sin \frac{\alpha}{2}\right),$$
since
$$\left(\cos \frac{\alpha}{2} + i \sin \frac{\alpha}{2}\right)^2 = \cos \alpha + i \sin \alpha,$$
i.e. $\sqrt{p + iq} = a + ib$ where $a = \sqrt{r}.\cos \dfrac{\alpha}{2}$, $b = \sqrt{r}.\sin \dfrac{\alpha}{2}$, and α is determined by $\tan \alpha = \dfrac{p}{q}$.

Here, too, $a + ib$ can be determined algebraically in the following way:

It can be shown that if $x + iy = l + im$,

then
$$x = l \text{ and } y = m.$$

Now if
$$\sqrt{p + iq} = a + ib,$$
$$p + iq = a^2 - b^2 + 2abi;$$
i.e.
$$a^2 - b^2 = p,$$
$$2ab = q,$$
and from these equations a and b can be determined in terms of p and q.

These algebraical corroborations confirm the validity of the Argand diagram.

EXAMPLE 3.—To express $\log (p + iq)$ in the form $a + ib$.
$$\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

Substituting $(p - 1) + iq$ for x, we obtain an expression of the form required.

In the same way $\sin \theta$ can be expanded in powers of θ; and if, for θ, $p + iq$ is substituted, an expression of the form $a + ib$ is obtained for $\sin (p + iq)$.

Finally, we must not imagine that all this investigation of i is merely a form of mathematical amusement; i plays a part in investigations that lead to results of practical importance—a part it could not have played if mathematicians had been deterred by its apparent meaninglessness.

We shall sketch in outline some of these results and, in doing so, we shall consider all angles to be measured in circular or radian measure:
$$(\cos \alpha x + i \sin \alpha x) = (\cos x + i \sin x)^{\alpha}.$$

Put $x = 1$,
$$\cos \alpha + i \sin \alpha = (\cos 1 + i \sin 1)^{\alpha}$$

For $\cos 1 + i \sin 1$ put c,

$$\cos \alpha + i \sin \alpha = c .$$

It can be shown that $c = e^i$ where e is the sum to infinity of

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

and $$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad (\textit{See} \text{ pp. } 180\text{–}1.)$$

$$\therefore \quad \cos \alpha + i \sin \alpha = e^{\alpha}$$

$$= 1 + i\alpha - \frac{\alpha^2}{2!} - \frac{i\alpha^3}{3!} + \ldots$$

Equating real and imaginary parts as in Example 2,

$$\cos \alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \ldots$$

$$\sin \alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \ldots$$

From these series trigonometrical tables can be calculated. (Originally they were not so calculated.)

Again, as Euler established,

$$e^{i\pi} = \cos 2\pi + i \sin 2\pi$$

$$= -1,$$

i.e. $$e^{i\pi} + 1 = 0,$$

an equation of simple form which involves three of the most interesting numbers in mathematics, and which has provided a step in the proof that π is transcendental (*see* p. 62).

This discussion of complex numbers is a warning that the results of mathematical analysis must not be dismissed because at first sight they seem to have no real meaning. Even if upon further investigation they appear to lead to results of no practical importance, they widen the horizon of algebra. But it is unlikely that such investigation will not sooner or later bring a reward of practical utility. In all the sciences it is common experience that investigations undertaken by specialists for their intellectual delight do eventually confer on the world, not only an accession of knowledge, but also an endowment of usefulness.

CHAPTER XV

GENERALIZATION AND EXTENSION

L'Algèbre est généreuse ; elle donne souvent plus que l'on ne lui demande.
—D'ALEMBERT.

IT is noticeable early in a boy's mathematical work that algebra generalizes his experience in arithmetic, and that it thereby introduces, and demands an interpretation of, numbers that would not arise in arithmetic. The training of the mathematician leads him to deal with the general rather than the particular, and to generalize the restricted or apparently restricted.

He prefers to deal with a problem in x, a, b, rather than a problem involving, say, 3, 7, 15 ; he extends his investigations

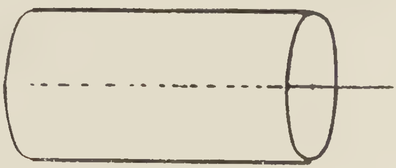


FIG. 204

of the triangle to the polygon, of plane figures to solid ; using the convention of sign, he need not differentiate between sums and differences, between internal and external division. Thus the angle-sum of a triangle is 2 right angles ; this leads to the determination of the angle-sum of an n-gon, which is $(2n - 4)$ right angles, and the triangle is then perceived to be a particular case of a polygon. Pythagoras' Theorem leads to the more general theorems comprised under the trigonometrical formula $c^2 = a^2 + b^2 - 2ab \cos C$, and is then seen to be a particular case of this general formula, cos 90° being 0. The locus of points in a plane equidistant from two given points A and B is the right bisector of the line AB ; the locus of points in space equidistant from A and B is a plane perpendicular to AB and bisecting it. The locus of points in a plane at a constant distance from a given straight line XY is two straight lines ; the locus of points in space at a constant distance from XY is the surface of a cylinder. In these cases the 3-dimensional locus is more tangible and more satisfying than the plane locus.

[Consider in the same way the locus of points in space equidistant from two given intersecting straight lines.]

The co-ordination of sum and difference has already been treated in the chapter on Negative Magnitudes.

Let us now consider some other cases.

We have seen (p. 195) that if $\overline{AB}$ represents the displacement of B from A, then for points on a straight line $\overline{AB} + \overline{BC} = \overline{AC}$.

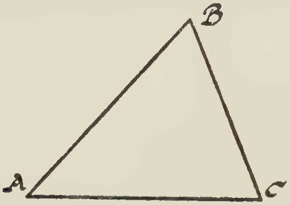


Fig. 205

This extended to points in a plane is the foundation of vector geometry, with its important applications to mechanics. It can also be extended to points in space; for any points A, B, C, D we have $\overline{AB} + \overline{BC} + \overline{CD} + \overline{DA} = 0$.

The formula $a^2 = b^2 + c^2 - 2bc \cos A$, itself an extension of Pythagoras' Theorem to any triangle, can be extended to any polygon.

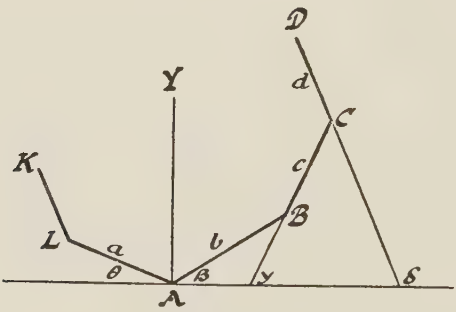


Fig. 206

Let ABCD . . . KL be a polygon. Through A draw two rectangular axes AX and AY. Take the lettering of the figure and let $\angle (b, c)$ be taken to mean the internal angle of the polygon included between the sides b and c.

Then projecting along AX, and keeping the conventions of sign,

$$-a \cos \theta = b \cos \beta + c \cos \gamma + d \cos \delta + \ldots\ldots \quad \text{(I)}$$

and projecting along AY

$$a \sin \theta = b \sin \beta + c \sin \gamma + d \sin \delta + \ldots\ldots \quad \text{(II)}$$

Squaring and adding

$$\begin{aligned} a^2 &= b^2 + c^2 + \ldots \\ &\quad + 2bc\,(\cos\beta\cos\gamma + \sin\beta\sin\gamma) + 2bd(\cos\beta\cos\delta + \\ &\qquad \sin\beta\sin\delta) + \ldots \\ &= b^2 + c^2 + \ldots \\ &\quad + 2bc\cos(\gamma - \beta) + 2bd\cos(\delta - \beta) + \ldots \\ &= b^2 + c^2 + \ldots \\ &\quad - 2bc\cos\angle(b, c) - 2bd\cos\angle(b, d) - \ldots \end{aligned}$$

The triangle formula is a particular case of this, all the sides except a, b, and c being zero.

To square (I), an acquaintance is necessary with the formula for $(a + b + c + \ldots)^2$, an extension of $(a + b)^2$, and as this has an interest of its own, it deserves some consideration.

By multiplication

$$\begin{aligned} (a + b + c + d + \ldots)^2 \equiv a^2 + {} & b^2 + c^2 + d^2 + \ldots \\ & + 2ab + 2ac + 2ad + \ldots \\ & \qquad + 2bc + 2bd + \ldots \\ & \qquad\qquad + 2cd + \ldots \\ & \qquad\qquad\qquad + \ldots \end{aligned}$$

Using the same sort of procedure as for the geometrical representation of the identity $(a + b)^2 \equiv a^2 + b^2 + 2ab$, we get Fig. 207, which establishes the general identity.

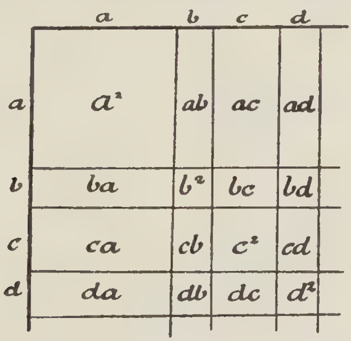


FIG. 207

Now, reading from the top left-hand corner of Fig. 207, the squares are

(1) a^2

(2) $\left.\begin{matrix} a^2 + b^2 \\ + 2ab \end{matrix}\right\}$, i.e. $(a + b)^2$

(3) $\left.\begin{matrix} a^2 + b^2 + c^2 \\ + 2ab + 2ac \\ + 2bc \end{matrix}\right\}$, i.e. $(a + b + c)^2$

Each gnomon (Gk. *a carpenter's rule*), i.e. the ⅃-shaped strip which added to one square produces the next, is the representation of the quantities of one of the columns in the algebraical identity.

Thus the gnomon which added to the first square makes the second represents $b^2 + 2ab$.

In the particular case where $a = b = c = d \ldots$ this arrangement in columns in algebra or in gnomons in geometry gives us that

$$1 + 3 + 5 + \ldots \text{ to } n \text{ terms} = n^2.$$

A fanciful analogy to a table of football fixtures, where a, b, c, d may be taken to represent teams, the rectangle ab to represent the match between a and b on a's ground, and ba the match between a and b on b's ground, may serve as a mnemonic.

In school geometries it is proved that in a cyclic quadrilateral the sum of opposite angles is 2 right angles. Is this a special case of a more general theorem? The consideration of the question is bound up with an analogous one—the property of a quadrilateral or polygon in which a circle can be described touching all the sides. Now the centre of a circle which passes through the *angular* points of a triangle is obtained by the intersection of the right *bisectors of the sides*; the centre of the circle which touches the *sides* of a triangle is obtained by the intersection of the *bisectors of the angles*. The analogy is based on an interchange of *angles* and *sides*.

Consider a quadrilateral ABCD in which a circle is inscribable.

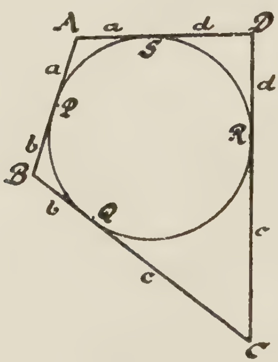


FIG. 208

Let P, Q, R, S be the points of contact.

Then since tangents to a circle from an external point are equal, using the lettering of the figure, we have

$$\begin{aligned} BC + DA &= (b + c) + (d + a) \\ &= (a + b) + (c + d) \\ &= AB + CD, \end{aligned}$$

i.e. the sum of one pair of alternate sides = the sum of the other pair.

This method of proof extends the property to a polygon of

$2n$ sides, i.e. the sum of one set of alternate sides = the sum of the other set.

And analogy suggests for the cyclic quadrilateral another form of enunciation, viz., that the angle-sum of one set of alternate angles = the angle-sum of the other set. This would be extended to a cyclic $2n$-gon as the sum of one set of alternate angles = the sum of the other set. But the analogy also suggests a general method of proof. In the theorem just proved *sides* are divided so that one part of one = the adjacent part of the next; in the cyclic polygon we should attempt (acting on the hint) to divide the angles so that one part of one = the adjacent part of the next.

Take, for example, a cyclic hexagon ABCDEF with O the circum-centre, and join OA, OB, etc.

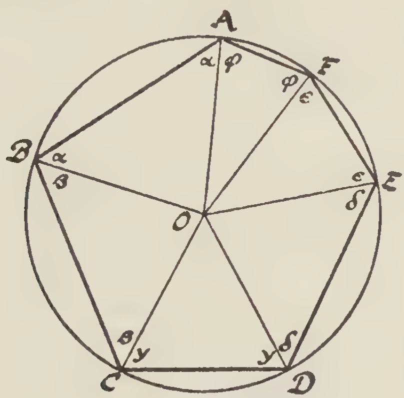


FIG. 209

The triangles thus formed are isosceles. Hence

$$\angle\text{s } B + D + \ldots = \alpha + \beta + \ldots + \varphi = \angle\text{s } A + C + \ldots,$$

providing that the number of sides is even.

The analogy can be carried slightly farther. In Fig. 208 both a's are adjacent to A, i.e. equal parts of sides are connected with an angle; in Fig. 209 both α's are adjacent to AB, i.e equal parts of angles are connected with a side.

Finally, one enunciation can be made to cover both theorems:

If a circle $\left\{\begin{array}{l}\text{passes through the angular points}\\ \text{touches the sides}\end{array}\right\}$ of a $2n$-gon, then the sums of alternate sets of $\left\{\begin{array}{c}\text{angles}\\ \text{sides}\end{array}\right\}$ are equal. It is to be noticed here as it has been elsewhere that the generalization of a theorem may demand a restatement of the enunciation.

Another case of some importance in geometry is that of the common chord of two circles. From the geometrical point of view

it would appear that for two circles to have a common chord they must intersect. From the algebraical point of view no such necessity exists.

Take two circles

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots \ldots \quad (1)$$

and

$$x^2 + y^2 + 2Gx + 2Fy + C = 0 \quad \ldots \ldots \quad (2)$$

x and y for their points of intersection are given by the solution of the equations (1) and (2).

Subtracting (1) and (2) we get

$$2(g - G)x + 2(f - F)y + c - C = 0 \quad \ldots \ldots \quad (3)$$

and (3), a real straight line, contains the points of intersection, and is therefore the common chord. The line is real even if the points of intersection are imaginary.

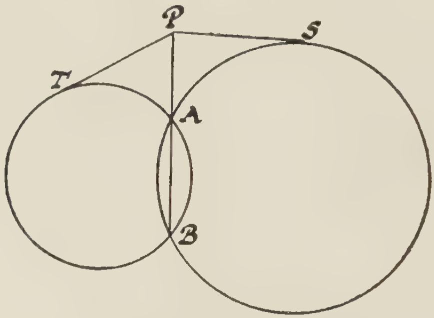


FIG. 210

Now, (3) can be regarded as the simplification of $x^2 + y^2 + 2gx + 2fy + c = x^2 + y^2 + 2Gx + 2Fy + C$; and if P (x,y) is any point on it, and PT the tangent to (1) it can be shown that $PT^2 = x^2 + y^2 + 2gx + 2fy + c$, i.e. the line (3) is the locus of points P from which tangents to the two circles are equal.

Geometrically, too, it can be shown that if P is a point on the common chord produced of two circles, the tangents PT and PS are equal. Substituting for the phrase "common chord" the phrase "the locus of points from which tangents to two circles are equal," we get a generalization which satisfies the algebraical analysis. We do not thereby escape the "imaginary." If the circles do not intersect, the "common chord" definition involves us in imaginary points of intersection, but all the tangents are real. If the circles do intersect, the points of intersection are real, but the tangents from A and B are zero, i.e. $x^2 + y^2 + 2gx + 2fy + c = 0$ for points on the circumference of (1), and the tangents from points between A and B are imaginary, i.e. $x^2 + y^2 + 2gx + 2fy + c$, which $= PT^2$, is negative. (*See* also p. 205.)

Fractional and Negative Number of Terms.—Consider the question, *How many terms of* $7 + 11 + 15$ . . . *must be taken to add up to* 52*?*

Using the formula for an A. P. we have

$$52 = \frac{n}{2}\{14 + 4(n-1)\},$$
$$2n^2 + 5n - 52 = 0,$$
$$(2n+13)(n-4) = 0,$$
$$n = 4 \text{ or } -6\tfrac{1}{2}.$$

Common sense and the ordinary definition of an A. P. would limit the solution to the positive integer 4, dismissing $-6\frac{1}{2}$ as meaningless, both because it is a fraction and because it is negative. But the experience of the algebra of indices might tempt us, working on the same lines, to see if there is not a reasonable interpretation for $-6\frac{1}{2}$ consistent with the behaviour of positive numbers.

From the laws of the series we deduce

$$S_n = \frac{n}{2}\{2a + (n-1)d\} \text{ for an A. P.}$$

$$S_n = \frac{a(r^n - 1)}{r - 1} \text{ for a G. P.}$$

Now these are algebraical functions of n which would have continuous graphs (in G. P. we will restrict ourselves to the cases where r is positive). We can find for any value of n a corresponding value of S_n. Is there for a given value of S_n a meaning to be given to the solution of the resulting equation in n?

Let us consider.

Take, first, what is in some ways the simplest A.P., 1, 3, 5, . . .

S_n here $= n^2$.

Suppose $\qquad S_n = 100.$
Then $\qquad 100 = n^2$
and $\qquad n = \pm 10.$

We can confirm by addition that $n = 10$ is correct, and our experience of negative magnitude as directed magnitude would lead us to inquire whether, by continuing the sequence backwards, ten terms thus obtained would add up to 100.

Write the series thus:

$$\ldots -5, -3, -1, 1, 3, 5, \ldots$$

It is clear that if 10 terms $\quad 1 + 3 + 5 + \ldots = 100,$
then $\quad$ 10 terms $\quad (-1)+(-3)+(-5) \ldots = -100,$
i.e. the numerical value 100 is accounted for; and we can account for the change in sign by remembering that we are adding one series (or writing it before addition) in a *backward* direction. We further note that the series begins not with the first term, but with the gap before it. Thus:

S_{-10} is not $\quad 1 + (-1) + (-3) + \ldots,$
but $\quad (-1) + (-3) + (-5) + \ldots$

Now let us see if these ideas will hold in any random case, say, S_{-3} of $5 + 8 + 11 + \ldots$

By formula $$S_{-3} = -\frac{3}{2}\{10 + (-3 - 1)3\}$$
$$= +3,$$
and we should expect that actual addition of 3 terms preceding the 5 would give -3. They are $2, -1, -4$, and the algebraical result is confirmed by actual addition.

In general, then, we should expect, if our conclusions are right, that

$$S_{-n} \text{ of } a + (a + d) + (a + 2d) + \dots$$
$$= -S_n \text{ of } (a - d) + (a - 2d) + (a - 3d) + \dots$$

Now

$$S_{-n} \text{ of } a + (a + d) + \text{etc.} = -\frac{n}{2}\{2a + (-n - 1)d\}$$
$$= -\left[\frac{n}{2}\{2(a - d) + (n - 1)(-d)\}\right]$$

i.e. $$= -S_n \text{ of } (a - d) + (a - 2d) + \text{etc.}$$

Thus we can interpret reasonably and concretely S_{-n} if n is a positive integer.

Applying the same considerations to a G. P. we should expect

$$S_{-n} \text{ of } a + ar + ar^2 + , \dots = -S_n \text{ of } \frac{a}{r} + \frac{a}{r^2} + \frac{a}{r^3} + \dots$$

$$\text{Now } S_{-n} \text{ of } a + ar + ar^2 + , \dots = \frac{a(r^{-n} - 1)}{r - 1}$$
$$= \frac{-a\left(1 - \frac{1}{r^n}\right)}{r - 1}$$
$$= \frac{-\frac{a}{r}\left(1 - \frac{1}{r^n}\right)}{1 - \frac{1}{r}}$$
$$= -S_n \text{ of } \frac{a}{r} + \frac{a}{r^2} + \frac{a}{r^3} + \dots$$

For fractional values of n, again try a simple arithmetical progression, e.g. to interpret $S_{2\frac{1}{3}}$ of 5, 8, 11.

$S_{2\frac{1}{3}}$ should be $> S_2$ but $< S_3$,

i.e. > 13 but < 24.

By formula $$S_{2\frac{1}{3}} = \frac{7}{6}\left(10 + \frac{4}{3} \times 3\right)$$
i.e. $16\frac{1}{3}$.

We get near to this by taking $S_{2\frac{1}{3}}$ to mean 1st term + 2nd term + $\frac{1}{3}$ of 3rd term; this gives $16\frac{2}{3}$

Every test based on this idea leads to a small difference between the result by addition and the result by formula. Now S_1 is not

$\frac{1}{2}S_2$, nor $\frac{1}{3}S_3$. If this were so, the formula for S_n would be a 1st degree function of n, whereas it is a 2nd degree function. In the same way $S_{\frac{1}{3}}$ should not be $\frac{1}{3}S_1$; for, if we divide S_3 into the three terms of which it is composed, we do not divide it into three equal parts, but into three parts which obey the law of the A. P. Similarly, to find $S_{\frac{1}{3}}$ we must divide S_1 into three parts which are not equal but obey the law of the A.P.

Now consider 5, 8, 11, each divided into three parts, so that the 9 resulting numbers are an A.P.

Thus if $5 = x + (x + y) + (x + 2y)$,
then $8 = (x + 3y) + (x + 4y) + (x + 5y)$,
and $11 = (x + 6y) + (x + 7y) + (x + 8y)$;
i.e. $5 = 3x + 3y$,
$8 = 3x + 12y$,
$11 = 3x + 21y$.

Solve any two: $y = \frac{1}{3}$, $x = 1\frac{1}{3}$, and this satisfies the third.

5 then is divided as the sum of $1\frac{1}{3}$, $1\frac{2}{3}$, 2,
8 do. $2\frac{1}{3}$, $2\frac{2}{3}$, 3,
11 do. $3\frac{1}{3}$, $3\frac{2}{3}$, 4.

And $S_{2\frac{1}{3}} = 5 + 8 + 3\frac{1}{3}$, the $3\frac{1}{3}$ being the result of a division according to the law of the progression, and this agrees with the result obtained from the formula.

In general, to find $S_{p/q}$ of a, $a + d$, $a + 2d$, we must divide a, $a + d$, $a + 2d$, each into q parts, so that all the resulting numbers form a new A.P.

Let $a = x + (x + y) + (x + 2y) + \ldots$ to q terms.

Then $a + d = (x + qy) + (x + \overline{q + 1}y) + \ldots$ to q terms.

Whence, by summation,

$$a = \frac{q}{2}\{2x + (q - 1)y\} \ldots\ldots\ldots\ldots (1)$$

$$a + d = \frac{q}{2}\{2(x + qy) + (q - 1)y\} \ldots\ldots\ldots (2)$$

By subtraction

$$d = q^2y \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{I})$$

and by substitution for y in (1), written in the form

$$a = \frac{q}{2}\{(2x - y + qy)\},$$

we get

$$a = \frac{q}{2}\left\{(2x - y) + \frac{d}{q}\right\},$$

i.e.

$$\frac{2a}{q} = 2x - y + \frac{d}{q},$$

i.e.

$$2x - y = \frac{2a - d}{q} \ldots\ldots\ldots\ldots\ldots (\mathrm{II})$$

Now S_p of $x + (x + y) + (x+2y) + \dots = \frac{p}{2}\{2x + (p - 1)y\}$

$$= \frac{p}{2}\{(2x - y) + py\},$$

and, substituting from (I) and (II),

$$S_p = \frac{p}{2}\left\{\frac{2a - d}{q} + \frac{pd}{q^2}\right\}$$

$$= \frac{p}{2q}\left\{2a + \left(\frac{p - q}{q}\right)d\right\}$$

$$= \frac{p}{2q}\left\{2a + \left(\frac{p}{q} - 1\right)d\right\},$$

i.e. S_p of the fractional parts of the new series

$= S_{p/q}$ of $a + (a + d) + \dots$

The same interpretation may be tried in the case of a simple G. P. and found to hold.

In general to find $S_{p/q}$ of $a + ar + ar^2 + \dots$

Let $a = x + xy + xy^2 + \dots$ to q terms,

$ar = xy^q + xy^{q+1} + \dots$ to q terms.

Whence, by summation,

$$a = \frac{x(y^q - 1)}{y - 1} \quad \dots\dots\dots\dots\dots \quad (1)$$

$$ar = \frac{xy^q(y^q - 1)}{y - 1} \quad \dots\dots\dots\dots\dots \quad (2)$$

By division $r = y^q$

and $y = r^{1/q}$ . . . . (I)

and substituting in (1) written in the form $a = \frac{x}{y - 1}(y^q - 1)$,

$$a = \frac{x}{y - 1}(r - 1)$$

i.e.

$$\frac{x}{y - 1} = \frac{a}{r - 1} \quad \dots\dots\dots\dots\dots \quad (II)$$

$$S_p \text{ of } x + xy + xy^2 + \dots = x \cdot \frac{(y^p - 1)}{y - 1}$$

$$= \frac{x}{y - 1}(y^p - 1),$$

and substituting from (I) and (II)

$$S_p = \frac{a}{r - 1}(r^{p/q} - 1),$$

and this is $S_{p/q}$ of the original series.

The important consideration here is that the extension to a

fractional number of terms depends on a fractional division, in which the fractional parts are not equal but themselves conform to the general law of the sequence.

The reader may apply these ideas to other series, e.g. to those on p. 270.

Asymptotes.—We shall conclude the chapter with an illustration taken from graphical algebra.

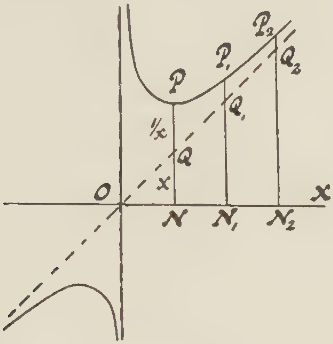


FIG. 211

To plot $$y = x + \frac{1}{x} \quad \ldots\ldots\ldots\ldots \quad (1)$$

we see that if we start with $y = x$, the addition PQ to be made for $\frac{1}{x}$ decreases as x increases, i.e. as $x \to \infty$, $PQ \to 0$. The line $y = x$ is an asymptote.

Again, as x becomes very small $\frac{1}{x}$ becomes very large, and as $x \to 0, y \to \frac{1}{0}$, i.e. $\to \infty$ and the line $x = 0$ is seen to be an asymptote.

Rewriting (1) as

$$(y - x)x = 1,$$

we see that each of the factors $y - x$ and x equated to zero gives an asymptote.

In general, if

$$L.M.N. \ldots = C \quad \ldots\ldots\ldots\ldots \quad (2)$$

where L, M, N are functions of x and y and C is independent of x and y, then $L = 0$, $M = 0$, $N = 0$, etc., are asymptotes of (2).

Our first experience of asymptotes is that they are straight lines; this need not be so.

Consider $$y = x^3 + \frac{4}{x - 2}.$$

Here $(y - x^3)(x - 2) = 4$, and the above argument will show that $y - x^3 = 0$ and $x - 2 = 0$ are the asymptotes.

With this knowledge a few plotted points help us to fix the curve of Fig. 212.

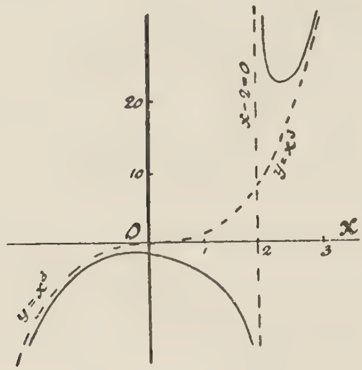


FIG. 212

Or take
$$y = x^2 \pm \frac{1}{4x^2},$$
i.e.
$$(y - x^2)x^2 = \pm \tfrac{1}{4}.$$

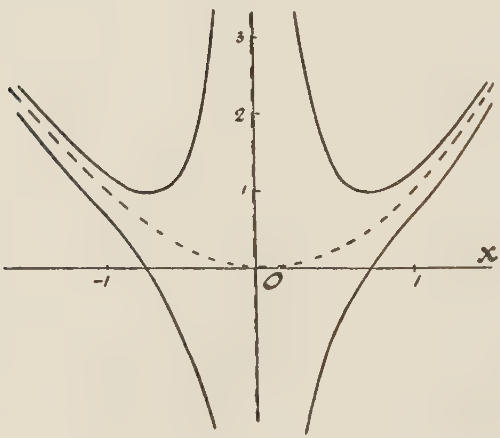


FIG. 213

The asymptotes are $y = x^2$, and two coincident lines $x = 0$. The curves are as shown in Fig. 213, the two inside branches

forming the curve $y = x^2 + \frac{1}{4x^2}$, the two outside forming the curve $y = x^2 - \frac{1}{4x^2}$.

To be able to determine the asymptotes is useful for getting rapidly a general idea of the shape of the curve; it is also useful as a foundation for actual plotting. Thus, when $y = x^2$ is plotted, the above curves can be drawn by adding values of $\pm \frac{1}{4x^2}$ to the ordinates.

CHAPTER XVI

SPECIAL CASES

If the instinct to generalize is part of a mathematician's equipment, it is also necessary to remember that the consideration of special cases may often be illuminating.

Thus if a theorem fails in one particular case it is not, as a general theorem, true; the application of a particular test may suffice for the rejection of a result due to wrong induction.

It is not uncommon for beginners to think that because the greatest side of a triangle is opposite the greatest angle, and vice versa, therefore the sides are proportional to the opposite angles. The case of any arbitrary right-angled or obtuse-angled triangle would be sufficient to show that the induction was false.

But the special case has positive as well as negative usefulness; it may tend to confirm a result and it may give useful guidance to a solution. Some examples have been given in the chapter on Symmetry to show that a special substitution may determine an unknown quantity, p. 137. Such a substitution may even supply the starting-point of a solution. Thus to factorize

$$b^2c + bc^2 + c^2a + ca^2 + a^2b + ab^2 + 3abc \quad \ldots\ldots \quad (1)$$

when $a = b = c = 1$, the expression becomes 9, and 9 factorizes as 3×3. This suggests that (1) may factorize into two factors each of three terms, and considering degree and symmetry we are led to try $(a + b + c)(a^2 + b^2 + c^2)$ or $(a + b + c)(bc + ca + ab)$, the latter of which is right. Or to factorize

$$a^3 + b^3 + c^3 - 3abc \quad \ldots\ldots\ldots\ldots \quad (2)$$

if c is made zero, the expression is reduced to a special case, $a^3 + b^3$, which factorizes as $(a + b)(a^2 + b^2 - ab)$; these factors must now be generalized by the inclusion of c, so as to satisfy the symmetrical property of (2), and we get $(a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$.

In the same way to factorize

$$x^2 - 6y^2 + 12z^2 - yz + 7zx - xy,$$

make $z = 0$, we have $x^2 - xy + 6y^2$, which $= (x - 3y)(x + 2y)$;
,, $x = 0$, ,, $-6y^2 - yz + 12z^2$, ,, $= -(3y - 4z)(2y + 3z)$;
,, $y = 0$, ,, $x^2 + 7xz + 12z^2$, ,, $= (x + 4z)(x + 3z)$.

Fitting these together so as to give two factors each containing an x term, a y term, and a z term, with the same coefficients as in the special cases, we have $(x - 3y + 4z)(x + 2y + 3z)$.

The reason for attempting to find factors in this way becomes clearer if x, y, z, each $= 0$, is substituted in the result,

$$x^2 - 6y^2 + 12z^2 - yz + 7zx - xy = (x - 3y + 4z)(x + 2y + 3z).$$

The method can be applied to an expression of the form $x^2 - 2xy - 3y^2 - 2x + 10y - 3$, if it is first made homogeneous by introducing z for unity. The expression becomes $x^2 - 2xy - 3y^2 - 2xz + 10yz - 3z^2$; and factorizes as

$$(x + y - 3z)(x - 3y + z),$$

i.e. as

$$(x + y - 3)(x - 3y + 1)$$

when 1 is substituted for z.

Or again, take a case in geometry. From a point A a straight line AQ is drawn to the circumference of a given circle. P is the mid-point of AQ; find its locus.

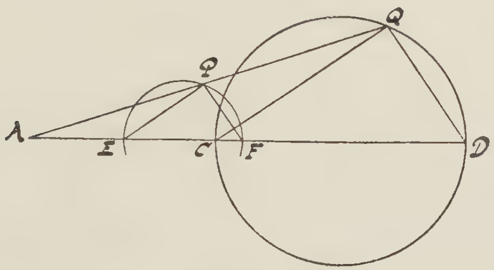


FIG. 214

The circle is symmetrical about any diameter.

Draw ACD to pass through the centre. Then E and F, the mid-points of AC and AD, are special positions of P. A few trial points suggest that the locus is a closed curve, and from considerations of symmetry EF is seen to be an axis of symmetry of the locus. As a circle is given, there is some inherent probability that the locus is a circle, and in any case a circular locus is the simplest to investigate. The locus is probably a circle on EF as diameter.

Take a position of Q and a corresponding P; join EP, PF and CQ, QD.

Then $\because$ AP = PQ and AE = EC $\therefore$ EP is parallel to CQ. Similarly PF is parallel to QD; $\therefore$ $\angle$ EPF = $\angle$ CQD = 90°, and this clinches the matter.

Sometimes the investigation of a special case is very much in the nature of an extension or generalization. Taking a triangle to be the figure bounded by the three straight lines which meet in three points, a special case would result from taking the three points as collinear. This is equivalent to assuming that the perpendicular

from A (Fig. 215) to the opposite side is of zero length. In such a triangle the sum of two sides = the third; one angle is 180°, the other two are 0°. Taking s for the semi-perimeter $\frac{1}{2}(a+b+c)$, $s-a$ is zero, and the area, $\sqrt{s(s-a)(s-b)(s-c)} = 0$. The radius of the inscribed circle, $\triangle/s = 0$; the circle itself is the point A; the tangent to it from C is CA, i.e. b; by formula it should be $= s-c$, and $s-c = b$, since $a = b+c$. Again, the centre of the circum-circle is the intersection of the right bisectors of BA and AC; these are parallel, therefore the circum-centre is at infinity; but the formula for the circum-radius $abc/4\triangle$ gives $R = \infty$. Again, the cosine formulæ reduce to $a^2 = b^2 + c^2 + 2bc$, $b^2 = c^2 + a^2 - 2ca$, $c^2 = a^2 + b^2 - 2ab$. These results

FIG. 215

are readily seen to be satisfactory. Other properties of a triangle may be tested, e.g. that the bisector of the $\angle$ BAC divides BC in the ratio of $c : b$.

Again, "If two angles of a triangle are equal, it is isosceles." Now $\angle$s B and C are equal, since each = 0°, but $b \neq c$. This apparent failure on the part of our special triangle to obey a theorem is explained by applying the general trigonometrical relation $b : c :: \sin B : \sin C$, i.e. $:: 0 : 0$ which is indeterminate.

Again, consider the regular n-gons inscribed in and circumscribed about a circle of radius r. Their perimeters are $2nr.\sin\frac{180°}{n}$, $2nr.\tan\frac{180°}{n}$, and their areas are $\frac{1}{2}nr^2.\sin\frac{360°}{n}$, $nr^2.\tan\frac{180°}{n}$.

These formulæ are true for all integral values of $n \nless 3$. There are two special cases to consider:

(1) When $n \to \infty$, the polygons converge to the circle, but as $n \to \infty$, $\sin\frac{180°}{n} \to \frac{\pi}{n}$, $\tan\frac{180°}{n} \to \frac{\pi}{n}$, and $\sin\frac{360°}{n} \to \frac{2\pi}{n}$, and each perimeter $\to 2\pi r$ and each area $\to \pi r^2$.

(2) When $n = 2$, the perimeters become $4r$ and ∞, and the areas become 0 and ∞. Now the perimeter of the inscribed polygon of two sides is, at any rate, a definite measurement, and it should be considered whether such a polygon may not reasonably be conceived. An inscribed polygon of n sides has n points on the circumference, dividing the circumference into n equal parts. A two-sided polygon would then have two points

A and B on the circumference, dividing the circumference into two equal parts, i.e. AB would be a diameter, and the inscribed 2-gon would be the figure bounded by AB and BA, its perimeter is twice the diameter, i.e. $4r$ and its area is 0. Given a regular inscribed polygon, the circumscribed polygon is bounded by the tangents at its angular points. The circumscribed 2-gon would then be the figure bounded by the tangents at A and B; its perimeter and area are both seen to be infinite.

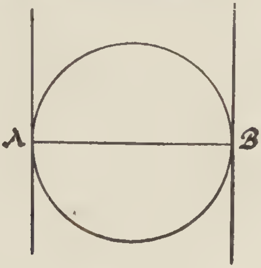


FIG. 216

Finally, it is to be noticed that before the introduction of algebra, the investigation of special cases in mathematics preceded and led up to the general: the investigation of the angles of a triangle preceded that of the angles of a polygon; the investigation of congruent triangles (triangles in which the common ratio of corresponding sides is 1 : 1) preceded and was necessary for the investigation of similar triangles (where the ratio is $m : n$); plane geometry preceded solid geometry, and so on. This mode of development is inherent in the synthetic character of pure geometry. In analytical or algebraical methods it is possible to deal with the general and take the particular as special cases; thus the investigation of the properties of the circle in algebraical geometry might start with either (1) the equation of the sphere or (2) the equation of a general conic; but though this would be possible, it would not, for teaching purposes, be expedient.

CHAPTER XVII

LIMITS

WHEN $x = 1$, the expression $\frac{x^2 + x - 2}{x^2 + 2x - 3} = \frac{0}{0}$. But $\frac{0}{0}$, taken independently of the function from which it is obtained, may have any value.

If we plot the function for any values of x from $-2\frac{1}{2}$ to 3, we get the graph shown in Fig. 217. If the function is continuous in the way indicated by the graph it will have one definite value,

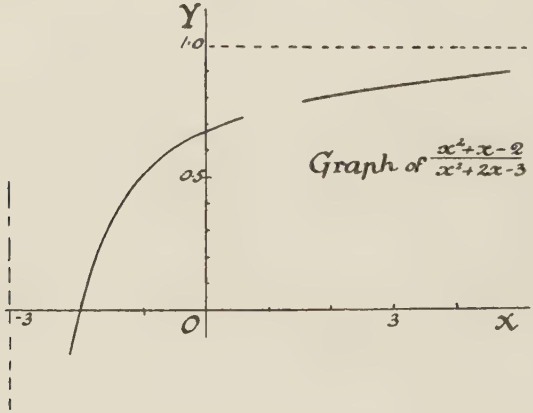


FIG. 217

and only one value, for $x = 1$, a value $> \frac{2}{3}$ (given by $x = 0$) and $< \frac{4}{5}$ (given by $x = 2$). By taking values of $x > 0$ but < 1, and > 1 but < 2 we can get closer approximations of the value to which the function tends as $x \to 1$.

To proceed more rapidly, substitute pairs of values ·9 and 1·1, ·99 and 1·01, ·999 and 1·001, etc., and tabulate as follows:

$f(\cdot9) = \cdot7436 = \cdot75 - \cdot0064$; $f(1\cdot1) = \cdot7561 = \cdot75 + \cdot0061$;
$f(\cdot99) = \cdot74937 = \cdot75 - \cdot00063$; $f(1\cdot01) = \cdot75062 = \cdot75 + \cdot00062$;
$f(\cdot999) = \cdot75 - \cdot000063$; $f(1\cdot001) = \cdot75 + \cdot000062$.

The actual division, which is not given here, is full of suggestion; the results are even more suggestive. They indicate that the nearer the value of x is to 1, the less is the error in considering the function as having the value ·75.

We say, then, that for the value $x = 1$ the limiting value of the function is $\cdot 75$. We write this statement

$$\mathrm{Lt}_{x \to 1} \frac{x^2 + x - 2}{x^2 + 2x - 3} = \frac{3}{4},$$

or,

$$\text{as } x \to 1, \frac{x^2 + x - 2}{x^2 + 2x - 3} \to \frac{3}{4}.$$

Dr. Whitehead, to whose "Introduction to Mathematics" in the Home University Library the reader is referred for a fuller discussion of this subject, uses this form of words, **The function $f(x)$ has the limit l at a value a of its argument, when in the neighbourhood of a its values approximate to l within every standard of approximation.**

In this case, for values of x in the neighbourhood of 1, $\frac{x^2 + x - 2}{x^2 + 2x - 3}$ approximates to $\frac{3}{4}$ within every degree of approximation, the error occurring in the 3rd, 4th, 5th, . . . significant figures as x differs from 1 by $\cdot 1$, $\cdot 01$, $\cdot 001$, . . .

The student is usually first introduced to the idea of a limit when he sums a geometrical progression "to infinity," i.e. he finds the limiting value of the sum of the series for every value of n however great.

Consider the simple case $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

The bookwork gives $2 - \frac{1}{2^{n-1}}$ for the sum of n terms.

The error in considering S_n as 2 is $\frac{1}{2^{n-1}}$.

This error is halved each time n is increased by 1. This can be shown in tabular form.

n	S_n	$2 - S_n$
1	1	1
2	$1\frac{1}{2}$	$\frac{1}{2}$
3	$1\frac{3}{4}$	$\frac{1}{4}$
4	$1\frac{7}{8}$	$\frac{1}{8}$
and so on.		

Now n can be chosen sufficiently large for the error $\frac{1}{2^n}$ not to affect any assigned significant figure, i.e. it can be made as small as we like by taking n large enough.

This can also be shown diagrammatically.

Take a rectangle ABCD, AB being 2 units of length and AD

being 1 unit. Bisect AB in U_1, U_1B in U_2, and so on, and draw U_1V_1, U_2V_2, U_3V_3, etc., parallel to AD.

Then AV_1, U_1V_2, U_2V_3, . . . represent u_1, u_2, u_3 . . . (1st, 2nd, 3rd . . . terms). AV_1, AV_2, AV_3 . . . represent S_1, S_2, S_3 . . . U_1C, U_2C, U_3C . . . represent the error in taking S_1, S_2, S_3 . . . as 2. As n is increased, this error $\rightarrow 0$. It is not enough to say that as n gets very great, S_n gets nearer to 2, therefore 2 is the limit; for we

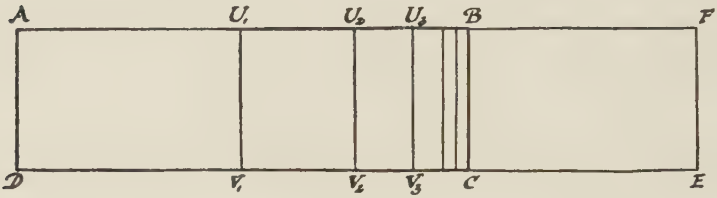


FIG. 218

could say truthfully enough that as n gets very great S_n gets nearer to 3, but 3 is not the limit. There is this difference: S_n can be made to differ from 2 by as small a difference as we please; S_n can never differ from 3 by less than 1, i.e. S_n is not equal to 3 for every degree of approximation.

In the diagrammatic representation make the rectangle AFED such that AF = 3 units of length; then AE represents 3. The lines U_1V_1, U_2V_2, etc., get nearer to BC; they also get nearer to EF; but the limiting position resulting from repeated bisection carried on an indefinite number of times is BC not EF.

In this case, then, where each fresh step of a process halves the error of the previous step, we see that we can make the error less than any assignable magnitude, however small, by repeating the process a sufficiently large number of times.

Now if each step had reduced the error to less than a half of the error of the previous step, we should have approached the limit more rapidly. This is applicable to the treatment of the area of a circle.

The Circle as the Limit of a Regular Polygon.—Let AB be a side of a regular n-gon inscribed in a circle, AC and CB sides of the regular $2n$-gon (Fig. 219).

At C draw the tangent LCM and complete the rectangle ABML. Draw CK perpendicular to AB.

In taking the area of the n-gon as the area of the circle, we have an error of n times the segment ABC.

In taking the area of the $2n$-gon as the area of the circle, we have an error of n times the sum of the segments ADC and CEB.

Now segment ADC $<$ △ ALC and ∴ $<$ △ AKC,
and ∴ segment ADC $< \frac{1}{2}$(ADC + △ AKC), i.e. $< \frac{1}{2}$ fig. ADCK ;
∴ sum of segments ADC and CEB $< \frac{1}{2}$ segment ACB,
i.e. the limit of the area of an inscribed n-gon when $n \to \infty$ is the area of the circle. This is contained in Euclid XII, 2.

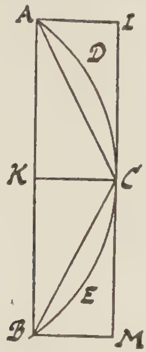


FIG. 219

Now let A and B be the points of contact of two consecutive sides of a regular circumscribed n-gon, AC and CB parts of the sides, K the mid-point of the arc AB, and LKM the tangent at K. Then ALMB is part of the perimeter of a regular $2n$-gon (Fig. 220).

The error in taking the area of the n-gon as the area of the circle is n times the fig. ADKEBC.

The error in taking the area of the $2n$-gon as the area of the circle is n times the sum of the figs. ADKL and KEBM.

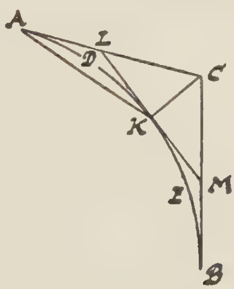


FIG. 220

In the △ LCM,

LM $<$ LC + CM ; ∴ LK $<$ LC.

But LK = LA ; ∴ LA $<$ LC ; ∴ △ AKL $<$ △ LKC.

But fig. ADKL $<$ △ AKL $<$ △ LKC and ∴ $< \frac{1}{2}$(ADKL + LKC),

i.e. $< \frac{1}{2}$ fig. ADKC ;

∴ figs. ADKL + KEBM together $< \frac{1}{2}$ fig. ADKEBC ;

∴ the area of the circle is the limit of the area of a regular circumscribed n-gon when $n \to \infty$.

Sum to Infinity (Dr. Nunn's phrase **Limiting-Sum** is preferable, and we shall use it here).—Returning to the G. P., consider such a case as

$$1 + \tfrac{1}{6} + \tfrac{1}{36} + \ldots$$

Summing arithmetically to 4-fig. and 6-fig. correctness,

4-fig.	6-fig.
1·	1·
·166 67	·166 66 67
27 78	27 77 78
4 63	4 62 96
77	77 16
13	12 86
2	2 14
1·200 00	36
	6
	1
	1·200 00 00

we get that the sum of the series is 1·2 to the degree of approximation to which we work. The student can test it for greater degrees of accuracy. The result agrees with the formula $S = \dfrac{1}{1 - \frac{1}{6}}$.

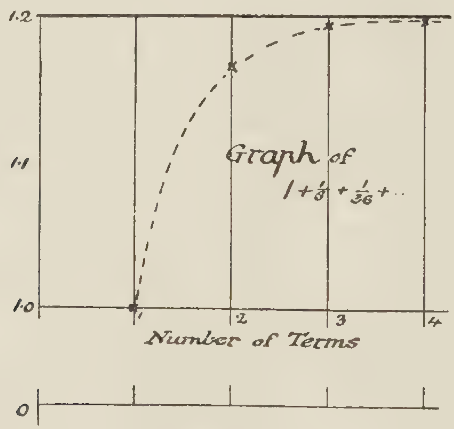


FIG. 221

Fig. 221 shows the graph of S_n; the parts of the ordinates produced which lie between the curve and $y = 1{\cdot}2$ show the error in taking S_n as 1·2. The line 1·2 is seen to be an asymptote to the graph, this condition in a graph indicating the approach to a limit.

The reader is referred to pp. 180–1, where e is obtained arithmetically as the limit of $\left(1+\frac{1}{n}\right)^n$ and also of S_n of $1+\frac{1}{1!}+\frac{1}{2!}+\ldots$ when $n \rightarrow \infty$.

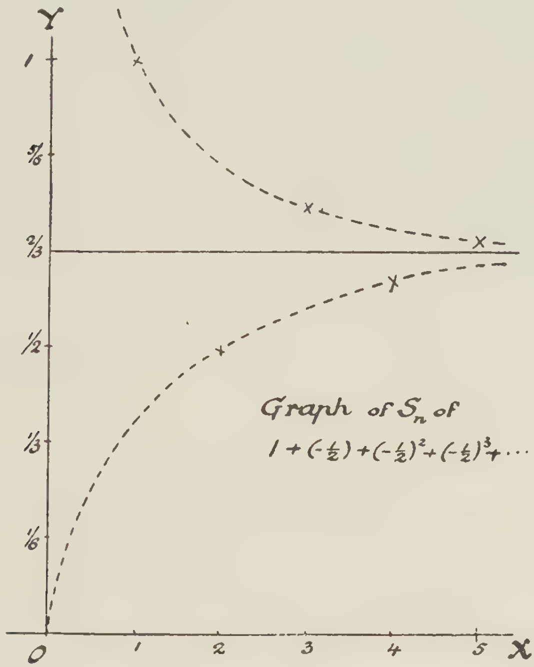


FIG. 222

For such a series as

$$1+\left(-\frac{1}{2}\right)+\frac{1}{2^2}+\left(-\frac{1}{2^3}\right)+\ldots$$

where the formula gives

$$S=\frac{1}{1+\frac{1}{2}}=\frac{2}{3},$$

the graph consists of two sets of points lying on two curves, to each of which the line $y=\frac{2}{3}$ is asymptotic.* Such a case where the limit always lies between two converging sets of values, one increasing and one decreasing, is even more convincing than when it is only the limit of one steadily increasing or steadily decreasing set.

We had such an example in regarding the circumference of

* Or it may be regarded as one set of points lying on a zigzag line.

the circle as the limit of perimeters of circumscribing and inscribed n-gons (p. 54).

When Brahmagupta (*see* p. 50) found $\sqrt{9{\cdot}65}$, $\sqrt{9{\cdot}81}$, $\sqrt{9{\cdot}86}$, $\sqrt{9{\cdot}87}$ for the area of inscribed n-gons, for values of $n = 12$, 24, 48, 96 sides, and jumped to the conclusion that π (the limiting value when $n \to \infty$) was $\sqrt{10}$, he was, of course, influenced by the importance of 10 as the base of arithmetical computation. If he had graphed his results (graphs were unknown to him) he would not have found any line parallel to OX to be clearly asymptotic to the graph. Certainly, if he had calculated circumscribed n-gons of the same number of sides he would have found values less than $\sqrt{10}$ for the area of those of 24 sides or more; i.e. $\sqrt{10r^2}$ would have been shown to be impossible as the common limit to which areas of inscribed and circumscribed polygons should converge.

We shall now discuss some cases of limits in which agreement

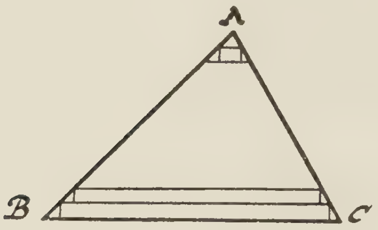


FIG. 223 (1)

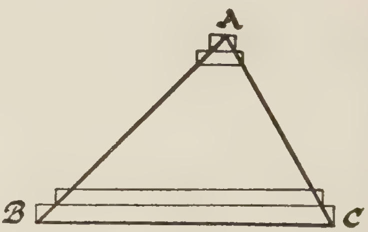


FIG. 223 (2)

with a known fact will support the results of mathematical procedure.

Consider a triangle ABC having a base BC of length a and an altitude of length h, and let $\angle$s B and C be acute. Let it be divided by straight lines parallel to BC into n strips, such that the altitude of each strip is $\frac{h}{n}$. Consider rectangles of altitude $\frac{h}{n}$ constructed on each dividing line, away from A in Fig. 223 (1), towards A in Fig. 223 (2).

By similar triangles the lengths of the dividing lines, starting from A, will be $\frac{a}{n}, \frac{2a}{n}, \frac{3a}{n}, \ldots$ and the area of the sum of the rectangles in Fig. 223 (1) will be $\frac{ah}{n^2} + \frac{2ah}{n^2} + \frac{3ah}{n^2} + \ldots$ to $(n-1)$ terms.

Since there are $n - 1$ rectangles, this

$$\text{total area} = \frac{(n-1)ah}{2n} = \frac{ah}{2} - \frac{ah}{2n} \quad \ldots\ldots\ldots\ldots \quad \text{(I)}$$

In Fig. 223 (2) there are n rectangles and the

$$\text{total area} = \frac{(n+1)ah}{2n} = \frac{ah}{2} + \frac{ah}{2n} \quad \text{. . . . . (II)}$$

In both (I) and (II) we have $\frac{ah}{2}$, which is independent of the number of strips we choose, but $\frac{ah}{2}$ is diminished in (I) and increased in (II) by $\frac{ah}{2n}$ (a function of n) which decreases if a larger number of strips is taken. Now, if we take twice as many strips we halve this function of n, and by continual subdivision we can make it less and less, and the mathematician says that in the limit when $n \to \infty$, $\frac{ah}{2n} \to 0$, and that in each figure the total area of the rectangles $\to \frac{ah}{2}$. He also says that in the limit when $n \to \infty$, the total area of the rectangles $\to$ the area of the triangle. This agrees with the common knowledge that the area of the triangle is $\frac{ah}{2}$.

The area of the triangle is here regarded as the limit to which two area-sums converge, one from within, one from without.

[Note :—The reader is advised to interpret and see the significance of $\frac{ah}{2n}$ for finite values of n.]

It must be borne in mind that however frequently subdivision is carried out, the limiting position cannot be attained. The step from the case where n is very large to the case where n is ∞ (as we say) cannot be actually made. But every case in which the mathematical treatment of limits leads to a confirmable result supports the validity of the mode of treatment.

Take, again, the recurring decimal $0{\cdot}\dot{0}\dot{1}$. This is a symbol for 0·010101 . . . the whole of which cannot be written down.

$$\text{Now it} = \frac{1}{100} + \frac{1}{100^2} + \frac{1}{100^3} + \dots$$

The sum of this series to n terms is

$$\frac{\frac{1}{100}\left[1 - \frac{1}{100^n}\right]}{1 - \frac{1}{100}} = \frac{1}{99}\left[1 - \frac{1}{100^n}\right]$$

$$= \frac{1}{99} - \frac{1}{99 \times 100^n}.$$

Thus $0{\cdot}\dot{0}\dot{1}$ is the remainder when from $\frac{1}{99}$, which is independent

of n, there is subtracted $\frac{1}{99 \times 100^n}$, and this fraction decreases as n increases, its limiting value being 0.

Now we can add as many terms as we like of the series $\frac{1}{100} + \frac{1}{100^2} + \ldots$, but we cannot add an infinite number; indeed the phrase "an infinite number of terms" is meaningless. Nevertheless we say that as $n \to \infty$, $\frac{1}{99} \times \frac{1}{100^n} \to 0$ and the sum of the series $\to \frac{1}{99}$.

We say, then, that $\frac{1}{99}$ is the limiting-sum of the series and $\therefore$ that $0{\cdot}\dot{0}\dot{1} = \frac{1}{99}$.

If we can reduce $\frac{1}{99}$ to the form $0{\cdot}01$ we show that our summation was justified, and the ordinary conversion of the vulgar fraction $\frac{1}{99}$ does give the decimal fraction $\cdot\dot{0}\dot{1}$.

Let us take another case: *What must be invested now at 4 per cent to yield annually, for ever, £4?*

Now, the present value of £4 one year hence is $\frac{4}{1{\cdot}04}$,

£4 two years hence is $\frac{4}{1{\cdot}04^2}$,

£4 three years hence is $\frac{4}{1{\cdot}04^3}$,

and so on.

Therefore, the amount invested must be the limiting-sum of $\frac{4}{1{\cdot}04} + \frac{4}{1{\cdot}04^2} + \ldots$

Now the quantities $\frac{4}{1{\cdot}04}$, $\frac{4}{1{\cdot}04^2}$, etc., can be evaluated to any degree of accuracy, and the results summed to any number of terms. To sum to any degree of accuracy, say 5-figure, would be a very laborious piece of computation, and even then the summation would have been effected only to a considerable number of terms, not to an infinite number.

We are confronted with the same difficulty as when we regard a circle as a polygon of an infinite number of sides (p. 234), or a triangle as the sum of an infinite number of rectangles (p. 238).

A certain geometrical procedure can be repeated, giving results which approach nearer to the limit. But it cannot be

repeated an infinite number of times. We meet the difficulty in the same way as before. We show that the *error* in the result for n repetitions of the procedure, i.e. the difference between the results for n repetitions (where n is large) and a certain magnitude independent of n, which we call *the limiting-sum*, is negligible to any degree of approximation. The argument in this case leads to the formula $S = \frac{a}{1 - r}$ for the sum to an infinite number of terms of a G.P. And this gives:

$$\frac{4}{1 \cdot 04} + \frac{4}{1 \cdot 04^2} + \ldots = \frac{\frac{4}{1 \cdot 04}}{1 - \frac{1}{1 \cdot 04}} = \frac{4}{\cdot 04} = \pounds 100.$$

And, of course, we know without any consideration of an infinite number of terms of a G.P. that £100 invested at 4 per cent per annum will give an annual return of £4 for ever. Thus the argument involved in obtaining the formula $S = \frac{a}{1 - r}$ is confirmed by simple considerations of quite another kind.

The arithmetical way of finding the limiting-sum has already been exemplified in the case of π (pp. 57–59) and e (pp. 180–1).

The practical importance of the arithmetical method is to be found in the evaluation of such numbers as logarithms, sines, and cosines. They can all be obtained from the summation of series; in general their value cannot be found exactly, but it can be found to any degree of accuracy required, i.e. their values as given in tables are the sums of certain series correct to so many figures or decimal places.

Recurring Continued Fractions.—There is a kind of fraction which has no exact value but which may be regarded as in some ways analogous to an infinite series. It is an infinite continued fraction.

We have already seen (p. 56) that one of the earliest definite expressions for π was in the form of such a continued fraction.

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \ldots}}}}$$

[It is usually written $\pi = \frac{4}{1+}\ \frac{1^2}{2+}\ \frac{3^2}{2+}\ \frac{5^2}{2+}\ \ldots$]

Any square root can be expressed as an infinite continued fraction, which will possess another property—that of recurrence—making it analogous to a recurring decimal.

A simple case is given, for $\sqrt{2}$:

$$1 < \sqrt{2} < 2.$$

Then
$$\sqrt{2} = 1 + \frac{\sqrt{2} - 1}{1}$$
$$= 1 + \frac{\sqrt{2} - 1}{1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1}$$
$$= 1 + \frac{1}{\sqrt{2} + 1}$$
$$= 1 + \frac{1}{2 + \frac{\sqrt{2} - 1}{1}},$$

and by repeating the process of rationalizing the numerator we find

$$\sqrt{2} = 1 + \frac{1}{2+}\;\frac{1}{2+}\;\frac{1}{2+}\;\ldots$$

a recurring infinite continued fraction.

Now
$$1 + \frac{1}{2} = 1{\cdot}5$$
$$1 + \frac{1}{2+}\;\frac{1}{2} = 1{\cdot}4$$
$$1 + \frac{1}{2+}\;\frac{1}{2+}\;\frac{1}{2} = 1\,\frac{5}{12} = 1{\cdot}41\dot{6}$$

By bringing in a greater number of terms we get successive approximations to $\sqrt{2}$:

$$1,\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \frac{41}{29},\ \frac{99}{70},\ \text{and so on;}$$

or expressed decimally to four places:

$$1,\ 1{\cdot}5,\ 1{\cdot}4,\ 1{\cdot}4167,\ 1{\cdot}4141,\ 1{\cdot}4143.$$

These results converge alternately from above and from below to a true value for $\sqrt{2}$.

If they are graphed $y = \sqrt{2}$ will appear asymptotic to two converging graphs, as in other cases already dealt with.

The theory of continued fractions is not regarded as belonging to the domain of elementary school mathematics. But this example is included here partly because it exhibits a surd as the limit of a converging series of definite form, partly because this form presents in vulgar fractions an analogy to the recurring decimal fraction, and partly because of the oscillatory behaviour of the convergents in approaching the limit.

CHAPTER XVIII

CONVERSES

In geometry a theorem is frequently followed by its converse, and usually this converse is true. Thus the proposition, *If two sides of a triangle are equal, the angles opposite those sides are equal,* is followed by the converse to this theorem, *If two angles of a triangle are equal, the subtending sides are equal,* which comes immediately after; and is true.

In this case there is a single premiss and a single conclusion, and there is only one converse. The theorem may be stated thus: "If $a = b$, then A = B," and the converse, "If A = B, then $a = b$." This is a common type.

There are theorems in which there is more than one premiss, and in that case there will be more than one converse. Consider the theorem, "In a circle the right bisector of a chord passes through the centre." It states that if a line (1) bisects a chord of a circle and (2) is perpendicular to it, then (3) it is a diameter. Three things are involved—bisection, perpendicularity, and the diametral property; given two of them, the third follows. In the theorem given (1) and (2), then (3) follows; the converses are two: first converse, given (1) and (3), then (2) follows; second converse, given (2) and (3), then (1) follows. These two converses are true.

The theorem might be stated so as to have one conclusion from three premisses:

"Given (1) a circle, (2) a line perpendicular to a chord, (3) that line bisecting the chord, then (4) the line is a diameter." Thus stated, we get the two converses of the geometry book already mentioned, but they would be stated thus: Given (1), (2) and (4), then (3) follows; and given (1), (3) and (4), then (2) follows. We also get a third converse: "Given a chord of a curve such that (2) a perpendicular, (3) bisecting it, (4) passes through a fixed point, then (1) the curve is a circle." This is true and easily proved.

In general, if n facts are connected so that one is a consequence of the remaining $n - 1$, then the theorem which states this has $n - 1$ converses, each of the remaining $n - 1$ facts being a deduction from all the rest.

It is often assumed that if a theorem is true its converse is true. Here is a typical case of a converse that is not true; starting with the proposition (assumed true) that "All boys who go to Sunbury School wear black jackets," we get the converse, "If

a boy wears a black jacket, he goes to Sunbury School." The truth of the converse cannot be relied on, and this is because the original proposition is only a part of the truth ; it gives no information respecting the attire of boys who do not go to Sunbury School.

If the original proposition had stated that (1) all boys who go to Sunbury School wear black jackets, (2) those who do not go to Sunbury School do not wear black jackets, then the converse would be true, viz., that if a boy wore a black jacket he would go to Sunbury School, and that if he did not wear a black jacket he would not go to Sunbury School.

This would give a complete classification of all boys into two groups : in the first group, attendance at Sunbury School corresponding to the wearing of a black jacket ; in the second group, non-attendance at Sunbury School corresponding to non-wearing of a black jacket. No boy in either group, whether classified according to his school or his jacket, could appear in the other. The dividing line between the groups cannot be crossed, and the *reductio ad absurdum* proof of the converse becomes possible.

In mathematics the classification is usually in three groups, e.g. (1) greater than, (2) equal to, (3) less than—for magnitude ; (1) outside, (2) on the boundary, (3) inside—for position. Applying a three-group classification to boys with respect to (1) their school, (2) the colour of their jackets, we might state a theorem thus : " All boys who go to Sunbury School wear black jackets.
,, ,, ,, other Schools ,, other-coloured jackets.
,, ,, ,, no School ,, no jacket."

Here again, with a complete classification into groups sharply divided, the familiar *reductio ad absurdum* proof can be applied to establish the converse.

Consider in this light the theorem (Euclid I, 22) : " If D is a point within a triangle ABC, then ∠ BDC is greater than ∠ BAC." From this statement alone the converse cannot be proved (in point of fact, it is not true), because the theorem does not state the whole truth ; it tells us nothing of the magnitude of ∠ BDC when D lies (1) on the periphery of the triangle, (2) outside the triangle.

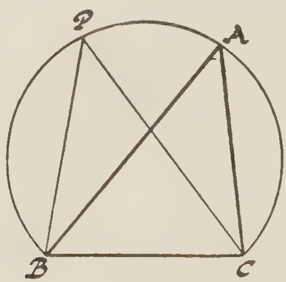


FIG. 224

If it could be proved that when D was on the periphery then $\angle$ BDC = $\angle$ BAC, and that when D was outside then $\angle$ BDC < $\angle$ BAC, then the converse could be proved. To investigate the truth of the converse we must therefore consider how the magnitude of $\angle$ BPC is related to the position of P.

Now the locus of points P such that $\angle$ BPC = $\angle$ BAC is the arc of a segment of a circle of which BC is a chord (Fig. 224).

Restricting ourselves to the A-side of BC we can classify all positions of D as being (1) outside the segment, (2) inside the segment, or (3) on the boundary arc, and we can prove that $\angle$ BDC < BAC for all outside positions, > BAC for all inside positions, = BAC for all boundary positions.

With the substitution of the segment BAC for the $\triangle$ ABC, as the area of reference, the theorem can now be stated as a threefold enunciation of which the converse is true.

The other part of Euclid I, 22, " If D is a point within a $\triangle$ ABC, then BD + DC < BA + AC," can be discussed in the same way (*see* p. 75). The reversibility of a step in algebra is a cognate question ; and most recent textbooks deal with it.

The threefold correspondence between position and magnitude appearing in the above cases is so common in geometry that it is an interesting and useful exercise to put enunciations in such a form as to emphasize it, the more so as, when this can be done, the converse of the proposition is true.

A few examples are given :

1. If A and B are two points, XY the right bisector of AB, and P any other point in the plane containing ABXY ;

$$\text{then PA}\left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\}\text{PB according as P is on }\left\{\begin{matrix}\text{the B-side} \\ \text{neither side} \\ \text{the A-side}\end{matrix}\right\}$$

of XY.

2. If ABC and DBC are two triangles, and AX and DY are perpendiculars on BC ;

$$\text{then } \triangle\text{ ABC}\left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\}\triangle\text{ DBC according as AX}\left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\}\text{DY.}$$

3. The circum-centre of a $\triangle$ is on the

$$\left\{\begin{matrix}\text{outside} \\ \text{periphery} \\ \text{inside}\end{matrix}\right\}\text{ of the }\triangle\text{ according as the }\triangle\text{ is}\left\{\begin{matrix}\text{obtuse angled} \\ \text{right angled} \\ \text{acute angled}\end{matrix}\right\}.$$

4. If a, b, c are the sides of a $\triangle$ and A, B, C the opposite angles,

$$a^2 - (b^2 + c^2)\text{ is}\left\{\begin{matrix}\text{positive} \\ \text{zero} \\ \text{negative}\end{matrix}\right\}\text{according as A is}\left\{\begin{matrix}\text{obtuse} \\ \text{right} \\ \text{acute}\end{matrix}\right\}.$$

5. If P is a point in the plane of a circle, the tangent from P

is $\left\{\begin{array}{l}\text{real}\\ \text{zero}\\ \text{imaginary}\end{array}\right\}$ according as P is $\left\{\begin{array}{l}\text{outside}\\ \text{on}\\ \text{inside}\end{array}\right\}$ the circumference of the circle.

We ought not to leave this topic without noting that the boundary position or magnitude is more restricted than the others ; thus there are many positive and many negative magnitudes, but only one zero ; for a given magnitude there are many inequalities, but only one equality. If P is a point on the line determined by points A and B, there are many positions of P in AB and many in AB produced, but only two boundary positions A and B ; if P is either inside or outside a circle, the range of its positions is an area, the boundary positions are restricted to a line.

CHAPTER XIX

INEQUALITIES

In a subject like mathematics, in which precision of thought and statement are fundamental, there might seem to be no place for the vaguely expressed relationships that are styled inequalities. In algebra, although inequalities have an important work to perform, they do not appear in the elementary parts of the subject. To the beginner, algebra presents the treatment of statements of equality, i.e. of equations. The equation $a = b + c$ is a precise statement; the inequality $a > b$ is not, it gives us no idea of the difference between a and b although it states that there is one. It is conceivable that in certain cases it may be easier to establish that $a > b$, and it may be more useful or necessary to do so than to establish $a = b + c$; in the early propositions of geometry this is found to be the case.

The examples that will be given of the relations connecting the elements of a triangle will illustrate this point, and it will become clear that in a deductive system it may be expedient to introduce theorems of inequality as steps in the process that leads up to the statement of equality.

Thus Euclid proved that the exterior angle of a triangle is greater than either interior and opposite angle before he proved that the exterior angle was the sum of the other two. If he had been able to prove the later proposition independently he could have dispensed with the earlier one; but he used the earlier not only for the proof of some other theorems but as a step in the train of reasoning by which the later proposition was demonstrated. The inequality theorem was for him indispensable to prove the equality theorem.

The theorem that $a + b > c$, where a, b, c are the lengths of the sides of a triangle, Euclid proved early. The precise relations which connect a, b, and c are neither so simple to express nor so easy to prove; the extensions of Pythagoras' theorem which embrace them, and which may be written in the form of the equation

$$c^2 = a^2 + b^2 - 2ab \cos C \quad \text{. . . . .} \quad (I)$$

come much later; and the distance which separates them from the inequality theorem is eloquent of the ingenious and intricate character of the deductive system of geometry.

Usually if the equality could be established independently, the inequality would be very simply deduced from it. In this

case the deduction by geometrical methods would not be easy; but the use of trigonometry for the purpose leads to an interesting discussion of the formula (I).

Making the substitutions, $2\cos^2\frac{C}{2} - 1$ or $1 - 2\sin^2\frac{C}{2}$ for $\cos C$ we get

$$c^2 = a^2 + b^2 + 2ab - 4ab\cos^2\frac{C}{2} = a^2 + b^2 - 2ab + 4ab\sin^2\frac{C}{2},$$

i.e. $c^2 = (a+b)^2 - 4ab\cos^2\frac{C}{2} = (a \sim b)^2 + 4ab\sin^2\frac{C}{2}$;

and since all the magnitudes involved are positive, $a \sim b < c < a + b$.

Again, consider the theorem, *If two triangles have two sides of one equal respectively to two sides of the other, then the triangle which has the greater included angle has the greater third side, and vice versa.* In other words, if a and b are fixed, c increases with C, and C increases with c. But the theorem does not establish a connexion between the rate or manner of increase of the side c and the rate or manner of increase of the angle C. Simple cases would show that c and C are not connected by direct variation. For let ABC (Fig. 225) be an isosceles triangle, having CA = CB.

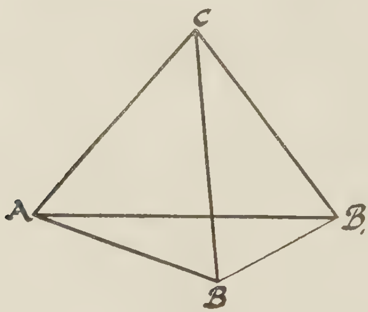


FIG. 225

Make $\angle BCB_1 = ACB$, and $CB_1 = CB$. Then $\angle ACB_1 = 2\angle ACB$.

But $AB_1 \neq 2AB$; for $BB_1 = AB$; $\therefore 2AB = AB + BB_1 > AB_1$.

The formula $c^2 = a^2 + b^2 - 2ab\cos C$ gives the law connecting c and C when a and b are fixed. Now, as C increases from 0° to 90°, $\cos C$ diminishes from 1 to 0, and c^2 increases from $a^2 + b^2 - 2ab$ to $a^2 + b^2$; as C increases from 90° to 180°, $\cos C$ diminishes from 0 to -1, and c^2 increases from $a^2 + b^2$ to $a^2 + b^2 + 2ab$.

That is, as C increases from 0° to 90° and on to 180°, c increases from $a \sim b$ to $\sqrt{a^2 + b^2}$ and on to $a + b$, these three special cases occurring when the $\triangle$ ABC is the straight-line $\triangle$ AB_1C, the

right-angled $\triangle AB_2C$, and the straight-line $\triangle AB_3C$ respectively. (*See* Fig. 226.)

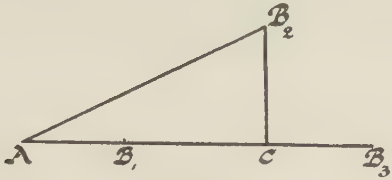


FIG. 226

The same theorem could have been investigated with the use of the forms $c^2 = (a + b)^2 - 4ab \cos^2 \frac{C}{2}$ or $(a - b)^2 + 4ab \sin^2 \frac{C}{2}$; $\cos \frac{C}{2}$ diminishing from 1 to 0 and $\sin \frac{C}{2}$ increasing from 0 to 1, as C increases from 0° to 180°.

We see, then, that parts of the precise and complete information contained in the equation $c^2 = a^2 + b^2 - 2ab \cos C$ are anticipated at various stages of our geometry. This recurrence of one property in various incomplete forms at various stages in the deductive sequence is interesting but not economical; it suggests that a freer use of algebraical or trigonometrical methods in our geometry would simplify its procedure.

In one case, at any rate, it happens that an equality theorem precedes an inequality theorem, and, as we might expect, appears in the proof of it. In a triangle if $a = b$, then $A = B$ and vice versa; this is an early theorem, from which it follows immediately that if $a \neq b$, then $A \neq B$ and vice versa; but the proof that if $a > b$, then $A > B$ and vice versa comes later. Here, again, even when we know the inequality theorems we do not know the exact relation which connects a, b, A and B. Many beginners imagine it to be simple proportion, i.e. $a : b : c$: A : B : C. That it is not so a simple special case, such as the triangle whose angles are 120°, 30°, 30°, would at once show. Trigonometry gives us the true relation, viz.,

$$a : b : c :: \sin A : \sin B : \sin C \quad \text{. . . . . (II)}$$

a statement of great importance and simple form, which Euclid's geometry never formulates.

And this relation will show that we must not conclude, though we might be inclined to do so, that if a and A are given, b will necessarily increase as long as B increases. In point of fact, if A is acute, then as B increases from 0° to 90°, b increases; but as B increases from 90° to (180° − A), b decreases; b is 0 when B is 0°, a maximum, a / sin A, when B is 90°, and = a when B is (180°−A).

This may be demonstrated geometrically.

Let BC be a; since $\angle$ A is given, the locus of the vertex A is

the circular arc CAB. When A coincides with C, $b = 0$; as $\angle$ B increases, b increases until CBA_2 is a right angle; then b is a diameter of the circle, and therefore a maximum. As $\angle$ B goes on increasing b diminishes until A coincides with B, when BA_4 becomes tangential and $c = 0$, $\angle$ CBA is $(180° - A)$, and CA coincides with BC, i.e. $b = a$.

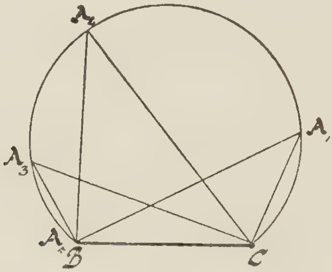


FIG. 227

The one algebraical inequality which occurs in elementary school mathematics is the fact that the arithmetical mean of two positive magnitudes is greater than the geometrical,

i.e. $$\frac{a+b}{2} > \sqrt{ab}.$$

This is only an incomplete form of the precise statement,

$$\left(\frac{a+b}{2}\right)^2 \equiv ab + \left(\frac{a \sim b}{2}\right)^2$$

In the same way, if a, b, c are positive and not all equal,

$$\frac{a+b+c}{3} > \sqrt[3]{abc}$$

is only an incomplete form of

$$\left(\frac{a+b+c}{3}\right)^3 \equiv abc + \frac{1}{54}\left\{\Sigma(7a+b+c)(b \sim c)^2\right\}.$$

In general, for n positive quantities not all equal

$$\frac{a+b+c+\ldots}{n} > \sqrt[n]{abc\ldots}$$

is only an incomplete form of

$$\left(\frac{a+b+c+\ \ldots}{n}\right)^n \equiv abc\ \ldots + \mathrm{R}.$$

But R, which is a function of a, b, c, and is positive, becomes more and more complicated as n increases, and it is usually not essential to know it.

Other algebraical uses of inequality are outside the scope of this work.

Finally, it must be remembered that the operations that

may be performed on equalities may not all be performed on inequalities.

Statements of equality have a property of reversibility in this way: If $a = b$, then $b = a$. It is a characteristic property; inequalities do not have it. Thus if $a > b$, it is wrong to say that $b > a$.

Again, operations that can be performed on equations cannot always be performed on inequalities. Euclid's axioms, *If equals be added to unequals, the wholes are unequal*, and the like, are examples of operations that can. A statement of equality remains a statement of equality if equals are added to the equal magnitudes; a statement of greater inequality remains a statement of greater inequality if equals are added to the two magnitudes.

The operations of addition, subtraction, multiplication by a positive number, division by a positive number, are operations permissible in the case of inequalities as well as of equalities.

But in the following cases, as may easily be seen, operations applicable in the case of equations lead to false results if applied to inequalities:

(1) If $\frac{a}{p} = \frac{b}{p}$, then $\frac{p}{a} = \frac{p}{b}$.
If $\frac{a}{p} > \frac{b}{p}$, then $\frac{p}{a} \not> \frac{p}{b}$.

(2) If $a = b$, then $a^2 = b^2$.
If $a > b$, then $a^2 > b^2$ is not true for all values of a and b.

(3) If $a = b$, then $pa = pb$.
If $a > b$, then $pa > pb$ is not true for all values of p.

(4) If $a = b$, then $p - a = p - b$.
If $a > b$, then $p - a \not> p - b$.

CHAPTER XX

DATA

In arithmetical problems it is recognized that for a determinate solution as many data are required as there are unknowns. Usually they will reduce to groups of simultaneous equations as many in number as the unknowns involved. But a restriction, such as a statement of inequality, may take the place of an equation, e.g. *A man who has only half-crowns owes* 3/6 *to a man who has only florins. How can the debt be paid?*

There are two unknowns—(1) the number of half-crowns to be paid, (2) the number of florins to be received as change. There is one definite fact given: the 3/6 is the difference between the value of the half-crowns and the value of the florins. In this case there is an indefinite number of solutions; one may be found by trial to be 3 half-crowns paid and 2 florins received in change; others may be found by adding any multiple of 4 to the 3 half-crowns and the same multiple of 5 to the 2 florins. The general solution is $3 + 4n$ half-crowns paid, $2 + 5n$ florins received, where n is any positive integer, and this leaves us with one unknown (viz. n).

If x is the number of half-crowns and y the number of florins, $5x - 4y = 7$; one equation with two unknowns. There is one equation too few for a determinate solution, and the equation is called an **indeterminate equation**. It can be represented by a graph which gives all solutions of the equation, and includes all solutions of the problem.

It is to be observed that while any value of x solves the equation giving a corresponding value of y, or while any value of y may solve it giving a corresponding value of x, we cannot take simultaneously any values of x and y to satisfy it—x and y are made dependent on each other, through the 3/6. This is seen on the graph. There is an indefinite number of solutions, but the indefinite number is restricted to the co-ordinates of points on the graph. The co-ordinates of points taken at random in the plane do not solve the equation. This condition of things corresponds, as we shall see, to the conditions for loci in geometry.

Now, if another fact is given, e.g. some relation of the numbers of the coins used—say, that 14 coins are used in all—we have a second equation and the solution is determined by

$$5x - 4y = 7 \text{ and } x + y = 14;$$

equations simultaneously true.

And the solutions are restricted to the co-ordinates of the points where the graphs intersect. Where the graphs are straight lines there will be only one solution, and if the value of either x or y is negative or fractional, there is no practical solution of the problem.

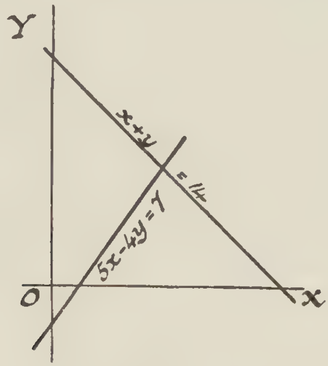


FIG. 228

But instead of a fact reducing to an equation we might have had the condition that the creditor has only 10 florins, and this restriction will reduce the indefinite number of solutions to a limited number;

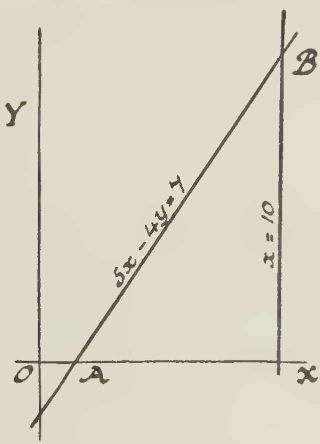


FIG. 229

it may even make a solution of the practical problem impossible. Here it gives us $2 + 5n \ngtr 10$, and for this n may be 0 or 1, i.e. there are two solutions: 3 half-crowns, 2 florins; and 7 half-crowns, 7 florins.

Graphically the solutions are restricted to that part of the

graph $5x - 4y = 7$ which lies on the negative side of $x = 10$, and only those points whose co-ordinates are both positive integers will satisfy the problem; that is, the solution is restricted to certain points on the line-segment AB.

In the solution of groups of simultaneous equations if there are n equations to determine n unknowns, the procedure is to eliminate one unknown—possibly by taking its value in terms of the others, as obtained from one equation and substituting it in the remaining $n - 1$ equations—and so to reduce to a group of $n - 1$ equations with $n - 1$ unknowns; then repeating the process to reduce to $n - 2$ equations with $n - 2$ unknowns, and so on till finally 1 equation is left with 1 unknown, giving a limited number of solutions corresponding to the degree of the equation (*see* pp. 172–3).

If only $n - 1$ equations are given for the determination of n unknowns, this process will give $n - 2$ equations for $n - 1$ unknowns, $n - 3$ equations for $n - 2$ unknowns, and so on, giving finally 1 equation for 2 unknowns, i.e. there will be an indefinite number of solutions; but the solutions will be restricted in this way, that if any value be taken for one unknown, the values of the others are dependent on it; for the substitution of that value in the equations leaves us with $n - 1$ equations involving $n - 1$ unknowns.

In geometry, to describe any figure a certain number of data are necessary; they may be measurements or conditions; if the necessary number are given there will be a limited number of solutions. Experience shows that to construct completely a triangle three data must be supplied. The trigonometrical formulæ confirm this; thus $c^2 = a^2 + b^2 - 2ab \cos C$ and $\dfrac{a - b}{a + b} = \dfrac{\tan \frac{A - B}{2}}{\tan \frac{A + B}{2}}$, each involve 4 elements; if 3 are known the 4th is determined by a single equation with 1 unknown.

The data are not necessarily elements of the triangle; one or more may be some other fact or condition about the triangle, e.g. in the problem to describe an isosceles triangle, having given the length of the base and the magnitude of the vertical angle, there are three data: (1) the length of the base, (2) the magnitude of the vertical angle, (3) the condition that two sides are equal. There is one solution.

There may be more than one solution, e.g. in the ambiguous case or problems like the following, *To draw a triangle given the lengths of two sides and the area.*

Again, to construct a circle in a certain position 3 data are necessary. The problem, *To describe a circle of given radius to touch two given circles*, may have 8 solutions or less. But there cannot be more; there cannot be an infinite number.

Now, if to describe a triangle or circle only two data are given, there will be one too few; the problem is analogous to that of solving two equations with three unknowns, which will reduce to that of solving one equation with two unknowns. The solution will be analogous to a graph in that the latter represents an indefinite number of points in restricted positions; the solution will be a locus of some undetermined point or the envelope of some undetermined line.

Thus, *To describe an isosceles triangle on a given base*, there is an indefinite number of solutions, but all vertices must be on the right bisector of the base; no point in the plane but outside that line can be the vertex. *To describe an isosceles triangle having an angle given in magnitude and position as the vertical angle*, the base may have an indefinite number of positions, but all will be parallel. *To describe a triangle on a given base and having a given vertical angle*, the vertex may have an indefinite number of positions but all will lie on one of two circular arcs.

To describe a circle of given radius to touch one given circle, the centre of the required circle may have an indefinite number of positions, but all lie on the circumference of one or of one of two circles. *To describe a circle to touch two given circles*, the centre of the required circle may have an indefinite number of positions; all will be on certain hyperbolas or ellipses.

In all these cases the absence of one of the data required for a determinate solution gives us a locus for a point that is to be determined; and just as a graph is itself a locus, so a geometrical locus results from the same conditions as give a graph in algebra, viz. the question of solving a problem requiring n data when only $n - 1$ are given.

If $n + 1$ data are given to determine n unknowns, one is superfluous and must be consistent with the others. Since the unknowns can be determined from n equations, when obtained they must satisfy the $(n + 1)$th, and the $(n + 1)$th equation is not an independent equation, but could be derived from the others.

Consider the three equations

$$3x - 2y = 0 \quad \ldots\ldots\ldots\ldots (1)$$
$$7x + 2y = 40 \quad \ldots\ldots\ldots\ldots (2)$$
$$x + y = 10 \quad \ldots\ldots\ldots\ldots (3)$$

involving only two unknowns; one is superfluous, but if the three are simultaneously true the solution of any pair must satisfy the third, or the solution of one pair must be the solution of any other pair. This is so.

The third equation is not independent but can be derived from the other two. From (1) and (2) $a(3x - 2y) + b(7x + 2y - 40) = 0$, for all values of a and b.

If this equation is to simplify to give (3), viz. $x + y - 10 = 0$,

$$3a + 7b = 1 \quad \text{(coefficients of } x\text{)}$$
$$-2a + 2b = 1 \quad \text{(coefficients of } y\text{)}$$
$$-40b = -10 \text{ (numerical term).}$$

These three equations are all satisfied by $b = \frac{1}{4}$, $a = -\frac{1}{4}$, and (3) could be obtained by subtracting (1) from (2), which gives $4x + 4y = 40$, and then dividing by 4.

[It would have been simpler to obtain (1) from (2) and (3) by elimination of the numerical term, but the more complicated method is given here because it is more general.]

The three equations (1), (2), and (3) can be represented by three straight lines, which must have a common point of intersection, whose co-ordinates (4, 6) satisfy all three equations.

If the graphs of three equations in x and y do not intersect in a common point, there is no common solution, and the equations do not form a group simultaneously true.

In order to determine whether the data given for a problem are sufficient, insufficient, or redundant, we must know how many are necessary. In geometry we may have to construct a figure in any position, i.e. we are concerned only with its size and shape, or we may have to construct it and place it in a certain position, i.e. we are concerned with size, shape, and position. We will consider both cases.

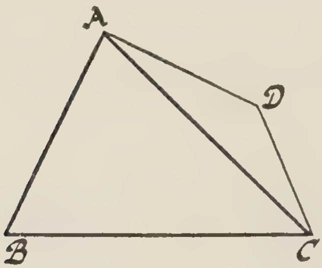


FIG. 230

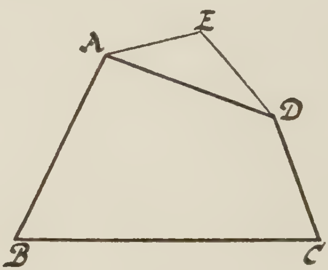


FIG. 231

We will assume that three data determine the size and shape of a triangle, and from this we will deduce the number of data required to determine the size and shape of a polygon. A quadrilateral ABCD can be regarded as consisting of two $\triangle$s ABC and ACD. To construct one (say ABC) three data are required, to construct the second, since AC is already given, only two other data are wanted. To construct a pentagon ABCDE we may construct a $\triangle$ ADE on the side AD of the already constructed quadrilateral ABCD; for the extra triangle two additional data are needed.

An n-gon can be regarded as being made up of $n - 2$ triangles. For the first triangle 3 data, and for each of the others

2 data, are needed, i.e. 2 for each triangle and 1 extra for the first, making altogether $2(n-2)+1$, or $2\iota - 3$ data.

Data for the Construction of a Plane Polygon of n sides Independently of Position.—To place an n-gon in a special position in a plane, say with reference to two axes of co-ordinates OX and OY, we require two data to fix each point A, B, etc. Therefore an n-gon can be constructed and placed in a plane if $2n$ data are given.

This result agrees with the one just obtained if 3 data (the difference between $2n$ and $2n - 3$) are required to fix the position of a polygon already constructed.

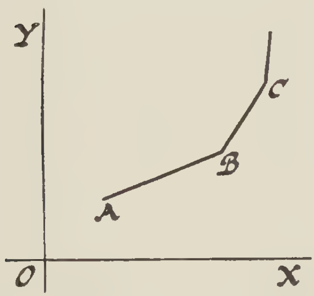


FIG. 232

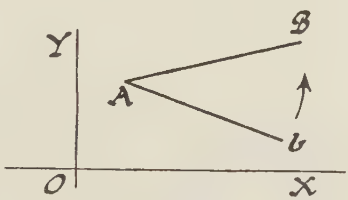


FIG. 233

To fix any point (say A) in its right position, 2 data are required, but AB may not be in the right direction (i.e. B may not be in its right position), let it have the position Ab (Fig. 233). It may then be turned through the $\angle\, b$AB until it is in the right position AB, and the rest of the polygon is then fixed. That is, 3 data (the two co-ordinates of A and the $\angle\, b$AB) were required to place a given polygon in a given position.

In dynamics it is considered that any motion in a plane can be analysed into (1) two separate motions of translation in directions parallel to two axes (say OX and OY), and (2) one motion of rotation about any axis perpendicular to the plane.

These motions correspond to **three degrees of freedom**; to prevent motion in a plane it is necessary to restrict these three degrees of freedom. Similarly, for equilibrium of a particle under coplanar forces the necessary and sufficient conditions of equilibrium are (1) that the algebraic sum of the components in each of two directions is zero, (2) that the algebraic sum of the moments about any one point is also zero.

These conditions suggest the following method of considering the fixing of the polygon (take a quadrilateral ABCD for convenience). (*See* Fig. 234.) It could be moved from any position *abcd* to the required position $\alpha\beta\gamma\delta$ by the following movements:

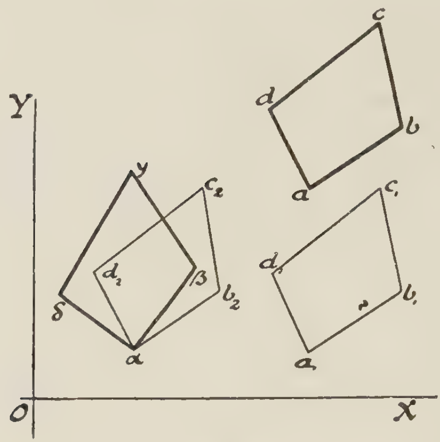


FIG. 234

(1) Parallel to OY to $a_1b_1c_1d_1$, where αa_1 is parallel to OX.
(2) Parallel to OX to $\alpha b_2c_2d_2$.
(3) By rotation about α to $\alpha\beta\gamma\delta$.

Data for an n-pointed polyhedron with triangular faces.—Now consider a figure in space. The tetrahedron is the unit; let it be ABCD. To determine the size and shape of BCD three

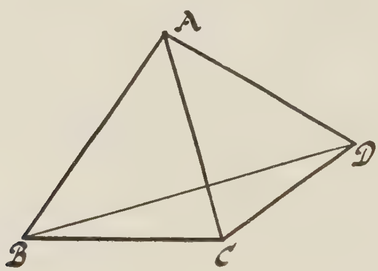


FIG. 235

data are required. To fix A relatively to BCD, it is sufficient to construct one of the sloping faces (say ACD), for which 2 data are required, and rotate it into position, for which one datum, i.e. the angle between the planes BCD and ACD, is required; i.e. to fix a new point 3 additional data are required. For a tetrahedron 6 data are required. Of this tetrahedron a plane quadrilateral ABCD, having the apex A in the plane BCD, is a special case, i.e. if a tetrahedron is given having the perpendicular from A to plane BCD of zero length, it reduces to a quadrilateral; but of the 6 data for the tetrahedron 1 has now been given, therefore 5 are

still required, the number already found necessary for the plane quadrilateral.

For a polyhedron of n angular points, having all its faces triangular, three data will be required for the triangle formed by three points of one face ; to fix each of the $n - 3$ remaining points 3 additional data are required, making altogether $3 + 3(n - 3)$, i.e. $3n - 6$ data.

Thus $3n - 6$ data are required to fix the shape and size of an n-hedron where all the faces are triangles.

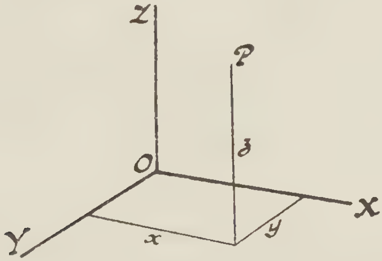


FIG. 236

To fix such a polyhedron in space, i.e. to find how many data determine its size, shape, and position, take the 3 planes of reference XOZ, XOY, YOZ, mutually at right angles. To fix a point 3 co-ordinates x, y, z are required. To fix n points $3n$ data are required.

This agrees with the above result if 6 data (the difference between $3n$ and $3n - 6$) are required to fix the position of a polyhedron already determined in size and shape.

Applied mechanics gives for a body in space six degrees of freedom, three of movement parallel to three axes, three of rotation about three axes ; and theoretical mechanics states as the necessary and sufficient conditions of equilibrium of a particle under any forces (1) the algebraical sums of forces in each of three directions = 0. (2) The algebraical sums of the moments about each of three axes = 0.

These considerations suggest not only that 6 data are necessary to place a given polyhedron in a required position but also a geometrical method of dealing with the problem.

To fix a given polyhedron ABC . . . in a required position $\alpha\beta\gamma$ . . . in space, it will be sufficient to fix one face ABC, just as to fix a polygon in a plane it was sufficient to fix one side.

To fix one angular point A at α 3 data are required. Suppose the position occupied by BAC to be $b\alpha c$ (Fig. 237). Rotation about $b\alpha$ will bring AC from the position αc into the plane of $\beta\alpha\gamma$. Let this operation put BAC into the position $b\alpha c_1$.

Let BAC be now rotated about αc_1, taking AB from the position αb to the position αb_1 in the plane of $\beta\alpha\gamma$.

BAC now occupies a position $b_1\alpha c_1$ in the plane of $\beta\alpha\gamma$, and one rotation in the plane about an axis through α perpendicular to the plane will take BAC from the position $b_1\alpha c_1$ to the required position $\beta\alpha\gamma$; that is to say, 3 rotations have sufficed (and 3 are necessary, though we have not proved this). Thus 6 data are

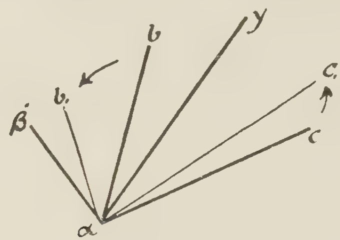


FIG. 237

required to fix the position, leaving $3n - 6$ data as necessary to determine the shape and size.

In any specific problem the polyhedron may have polygonal faces, in which case the number of data required will be less than $3n - 6$; allowance must be made for the conditions that various points are in the same plane as some set of three other points.

A Problem in Surveying.—In some problems there are implied conditions that must be taken into account. Thus one of the commonest types of a 3-dimensional problem in trigonometry deals with fixing a point P in space from observations made from the extremities of a base line AB, which we will take to be horizontal. In the surveying problem 5 measurements are taken, usually the length of AB, the bearings of P from A and B, and the angles of elevation of P at A and B, measured from the horizontal plane through A and B.

We are really concerned with a tetrahedron PABX (Fig. 238), where PX is perpendicular to the horizontal plane through AB. We ought to have then 6 data ($3 \times 4 - 6$) to find the dimensions of the figure. We have only 5, but we have assumed that PX is perpendicular to a plane, i.e. we have assumed that two angles PXA and PXB are right angles. Thus our data are really 7 in number, and 1 should be superfluous.

The remaining dimensions can be determined from the following data: the length of AB, the elevations of P from A and B, and *the difference of the bearings* of P from A and B, i.e. 4 data, which with the right angles PXA and PXB, implicitly assumed, make the 6 data required by the formula. For, letting PX be of length h, AX and BX can be determined in terms of h from the $\triangle$s APX

and BPX, and the angle AXB being known (the difference of the bearings) and AB being given, an equation can be written down for the $\triangle$ ABX, the solution of which gives h.

Or we may consider that A being a fixed point the line AP is determined in direction by the bearing and elevation of P from A. The plane PBX is determined by the bearing of P from B, and the lane PBX m eets the line AP in only one point ; i.e. the point P is

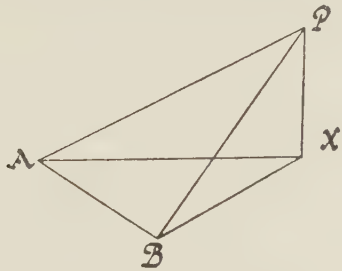


FIG. 238

determined independently of the elevation of P from B. Thus 1 datum is superfluous.

Or again, AP is determined in direction by the elevation and bearing of P from A ; BP is determined in direction by the elevation and bearing of P from B. But two straight lines in space will not necessarily meet, i.e. AP and BP will only meet if there is a relationship among the 4 data of bearings and elevations. In the practical surveying problem this relationship provides a check, i.e. the superfluous datum has a practical usefulness of great importance.

CHAPTER XXI

INDUCTION

Rule-of-thumb does not assist a boy "in inventing new things and practices . . . or in judging what comes before him," which should be the aim of education.—NEWTON : "SCHEME FOR SYLLABUS FOR CHRIST'S HOSPITAL."

THE mathematical method to which we are accustomed and to which textbooks generally conform is deductive and synthetic. In the course of mathematical discovery, isolated truths were obtained, perhaps by some flash of perception, perhaps as the result of pondering on a set of observations; proofs may have been delayed, or when obtained may have lacked rigour. Sooner or later it becomes the duty of mathematicians to marshal in an ordered sequence the knowledge that has been acquired so that the whole subject-matter is displayed as a series of steps in which each follows as a deduction from those that precede. In this straightforward sequence the proof of each step is built up, i.e. the method is synthetic. But the proofs may have been obtained by analysis, i.e. by working backwards until the result to be obtained was found to be dependent on some other result already established.

Thus it can be shown that the construction of a regular pentagon can be carried out if any isosceles triangle having an angle of 36° can be constructed. This in turn is found to depend on the division of a straight line in a certain way—often called "Medial Section," and this, in its turn, depends on the proposition that states that the square on a tangent to a circle from an external point is equal to the rectangle contained by the segments of any secant drawn from the point.

Here is a group of four propositions. The last mentioned, which becomes the first in the deductive arrangement, is one of considerable importance for other purposes in geometry. The medial section proposition is a particular case of a more general problem which is equivalent to solving a certain type of quadratic equation; the proposition itself is of little importance, as is the construction of the isosceles triangle, except for the fact that it is needed in order that the regular pentagon may be constructed; and the construction of the regular pentagon is chiefly important in Euclid's Elements because it is involved in the consideration of the regular solids, the dodecahedron and icosahedron—the culmination of the Elements.

The second and third propositions of the deductive sequence

are not therefore likely to enlist much interest until the fourth is reached, when at last it becomes clear what their function is. The analytic method would invest the subject with more interest, but it would involve a loss of compactness and mean frequent breaks in the straightforward line of thought.

Again, there are propositions that are so "obvious" to the pupil that proof seems to him to be unnecessary. His intuition tells him that they are right; but intuition is a fallible guide, e.g. it misleads perhaps the majority of beginners into believing that the area of any parallelogram is the product of the adjacent sides. Beginners, too, are generally found to believe that if corresponding sides of similar triangles are in the ratio $m : n$, their areas are in the same ratio. The results of intuition are not necessarily true, even when they are the intuitions of the majority; they must be proved before they can be safely accepted. But in the history of mathematics intuition has led to many important results, even when rigorous proof has for a time been dispensed with; and it is safe to say that most of us find the play of perception more fascinating than the march of deduction. When a pupil makes experiments, and his perception is free to find in the results some truth or law, he is treading in the steps and sharing the excitement of the discoverer. The educational value of this pleasure is lost if he is kept to the deductive method.

The method of jumping to a conclusion from certain observed or calculated facts is called **induction**, and mathematics has a method whereby it can sometimes be proved that if a law is true in a certain number of cases it is universally true.

The method is called **mathematical induction**. An example is given:

Consider the sequence of odd numbers 1, 3, 5, 7 . . . starting with unity.

We will use S_n as a symbol to denote the sum of n terms and u_n as a symbol to denote the nth term.

By simple addition

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + 3 = 4 \\ S_3 &= 1 + 3 + 5 = 9 \\ S_4 &= 1 + 3 + 5 + 7 = 16. \end{aligned}$$

Now 1, 4, 9, 16 . . . are $1^2, 2^2, 3^2, 4^2, \ldots$

We conclude that $S_n = n^2$. It is certainly true in all the cases we have tried; we can hardly imagine it to fail to continue to be true; but we need a proof.

To get S_{n+1} we must add to S_n the $(n + 1)$th term, which is $2n + 1$.

$$\therefore \quad S_{n+1} = S_n + 2n + 1;$$

and assuming

$$S_n = n^2$$

$$S_{n+1} = n^2 + 2n + 1 = (n + 1)^2.$$

Therefore if the law is true for n terms, it is true for $(n + 1)$ terms. But it was true for 4 terms, therefore it is true for 5, and

being true for 5 it is true for 6, and so on, i.e. it is universally true.

This particular series was summed by Pythagoras in the following way:

Take the square A to represent 1. Attach to it as in the figure a gnomon B (*see* p. 217), of 3 squares. This gnomon represents 3. The whole figure thus formed is a square and represents 2^2.

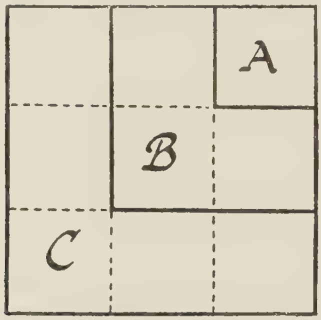


FIG. 239

Add a second gnomon C of 5 squares to represent 5. The resulting figure is a square and represents 3^2.

It is clear that this process produces at each stage a square, and that n areas made up of A + $(n - 1)$ gnomons, i.e. the area which represents S_n of $1 + 3 + 5 + \ldots$, is a square whose side has n units of length and which therefore represents n^2.

The law thus proved can be used to find the ratio of areas of similar triangles, provided that the ratio of corresponding sides is commensurable.

Take a $\triangle$ ABC; divide the base BC and the side BA each into m equal parts. Through the points of division $C_1, C_2, C_3 \ldots$ and $A_1, A_2, A_3 \ldots$ draw lines parallel to AC and BC, as in the figure.

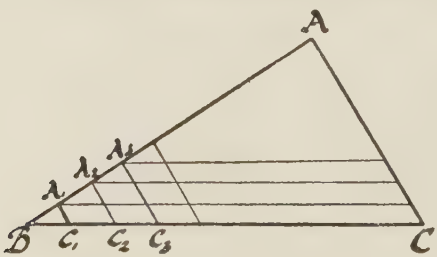


FIG. 240

Then let A_1BC_1 have an area $\triangle$. The strip $A_1C_1C_2A_2$ is made up of a triangle and a parallelogram; the triangle has an area $\triangle$, the parallelogram $2\triangle$, i.e. the strip has an area $3\triangle$.

The next strip is made up of one triangle and two parallelograms and has an area $5\triangle$, and so on.

Thus the area ABC is S_m of $\triangle + 3\triangle + 5\triangle + \ldots$, i.e. $m^2\triangle$.

In the same way a similar triangle whose base is $\frac{n}{m}$ of BC, i.e. n times BC_1, can be shown to have an area $n^2\triangle$, and the ratio of the areas of the triangles is proved to be $m^2 : n^2$.

It can similarly be proved by mathematical induction that if corresponding edges of similar tetrahedra are in the commensurable ratio $m : n$ their volumes are in the ratio $m^3 : n^3$.

Of course a law obtained by induction cannot always be demonstrated by mathematical induction. Consider the numbers of edges, vertices, and faces of a plane solid.

Make a table of these numbers for any solids:

Solid	E = No. of Edges.	V = No. of Vertices.	F = No. of Faces.
tetrahedron	6	4	4
cube	12	8	6
pyramid	8	5	5
octahedron	12	6	8
pentagonal prism	15	10	7
hexagonal pyramid	12	7	7

In each of these cases $E + 2 = V + F$.

This law is true for any other plane solid, regular or irregular, that you test. It can also be shown that if it is true for any given solid it is true for a solid that can be obtained from it by cutting off a corner by plane section.

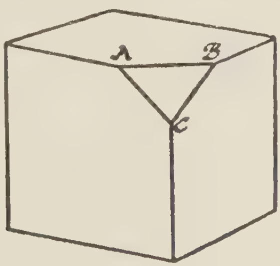


FIG. 241

Thus if a cube has a corner removed by the plane section ABC,

E is increased by 3 (AB, BC, CA)
F ,, 1 (ABC)
V ,, 3 (A, B, C), but it is also diminished by 1 (the lost vertex).

That is, E is increased by 3 and F + V is increased by 3, so that the law is unaltered.

This procedure can be generalized; so that if we could prove that all solids could be developed in this way from a limited number of fundamental solids, for which the law could be shown to be true, then the law would be universally true.

But this cannot be done. So that although the probability of the truth of the law is made greater, the demonstration of the law remains to be found. Euler's proof is given in most geometry books, and is an example of how a simple means may sometimes be found for settling a formidable difficulty.

The method of mathematical induction is especially useful in dealing with series. Thus take the case of the geometrical progression. First take the simplest series of this type, $1 + 2 + 4 + \ldots$ and tabulate in columns values of n, u_n, S_n, the values of S_n being obtained by addition.

n	u_n	S_n
1	1	1
2	2	3
3	4	7
4	8	15
5	16	31
6	32	63
n	2^{n-1}	
$n+1$	2^n	

It is seen in the numerical cases that

$$S_4 = u_5 - 1,$$
$$S_5 = u_6 - 1, \text{ etc.,}$$

and these results suggest the law

$$S_n = u_{n+1} - 1$$
$$= 2^n - 1.$$

Take $1 + 3 + 9 + \ldots$ We might expect S_n to be $3^n - 1$. Try it and tabulate for n, u_n, S_n, and $3^n - 1$.

n	u_n	S_n	$3^n - 1$
1	1	1	2
2	3	4	8
3	9	13	26
4	27	40	80
5	81	121	242
6	243	364	728

We see that the numbers in the S_n column are half the numbers in the $3^n - 1$ column, and we infer the law

$$S_n = \frac{3^n - 1}{2}.$$

Take $1 + 4 + 16 + \ldots$ and tabulate for n, u_n, S_n, $4^n - 1$;

n	u_n	S_n	$4^n - 1$
1	1	1	3
2	4	5	15
3	16	21	63
4	64	85	255
5	256	341	1023

and here we see that

$$S_n = \frac{4^n - 1}{3}.$$

We have now these results: If the constant ratio is 2, 3, 4, S_n is $\frac{2^n - 1}{1}$; $\frac{3^n - 1}{2}$; $\frac{4^n - 1}{3}$; and we infer that, for a ratio r, $S_n = \frac{r^n - 1}{r - 1}$.

To prove it, we proceed by mathematical induction

$$\begin{aligned} S_{n+1} &\equiv S_n + u_{n+1} \\ &= \frac{r^n - 1}{r - 1} + r^n \\ &= \frac{r^n - 1 + r^{n+1} - r^n}{r - 1} \\ &= \frac{r^{n+1} - 1}{r - 1}, \end{aligned}$$

which is of the right form; and we have proved that S_n of $1 + r + r^2 + \ldots$ is $\frac{r^n - 1}{r - 1}$.

But more interesting and far-reaching than the results of these examples are the developments which lead from the summation of the natural numbers to the statement and partial proof of the Binomial Theorem.

Some seventeenth-century mathematicians, including Newton and Pascal, were interested in problems of Probability; these involved problems of Permutations and Combinations. Let us take an easy one and watch a contemporary of Newton's, whom we will call Septimus, set about it. He wants to know in how many

ways a pair can be chosen from n rapiers, indistinguishable from one another. He takes a paper and puts on it a number of dots to represent the rapiers; he draws lines from any one dot to any other. Each line represents a combination of two rapiers, and the number of separate lines he can draw (no three dots being in one straight line) is the number of pairs of rapiers.

He makes diagrams of 2, 3, 4 . . . dots, draws and counts the numbers of lines, and tabulates results:

Dots	1	2	3	4	5	. . .
Lines	0	1	3	6	10	. . .

These numbers 0, 1, 3, 6 do not too obviously obey a law. But Septimus might count again in another way.

For 1 point there are no lines. An extra point gives the first line; a second extra point can be joined to each of the 2 points, giving 2 extra lines; a third extra point can be joined to each of the 3 points, giving 3 extra lines; and so on. And his results would be tabulated as shown here.

Points	Lines
1	0
2	$0 + 1$
3	$0 + 1 + 2$
4	$0 + 1 + 2 + 3$
5	$0 + 1 + 2 + 3 + 4$
n	$0 + 1 + 2 + 3 + \ldots + (n - 1)$
$n + 1$	$0 + 1 + 2 + 3 + \ldots + n$

He has found that the number of lines for $n + 1$ points is S_n of $1 + 2 + 3 + \ldots$

But there is still another way of finding the number of lines.

Thus, starting with $n + 1$ points, any one can be joined to each of n others. That is, through each point n lines can be drawn; therefore, through $n + 1$ points $n(n + 1)$ lines can be drawn. But each line has been counted twice in this procedure—once for each point through which it passes. So that the true number of lines is $\frac{n(n+1)}{2}$ and $\therefore$ S_n of $1 + 2 + 3 + 4 + \ldots = \frac{n(n+1)}{2}$.

Septimus has not only solved his problem but he has incidentally found out how to add n terms of the series $1 + 2 + 3 + \ldots$

He would probably check his solution in some random cases, and then set to work on the problem of finding how many sets of 3 can be chosen from n. Following a similar train of reasoning he would say: "For 3 dots I have one set; an extra dot can go with each pair of the original three to give an extra set; a second extra dot will go with each pair in the four to give an extra set." Thus he is dependent on the solution of his first problem for the numbers to be added in the second, and he might tabulate:

Dots	Pairs	Triads
1	0	
2	1	
3	3	$1 = 1$
4	6	$1 + 3 = 4$
5	10	$1 + 3 + 6 = 10$
6	15	$1 + 3 + 6 + 10 = 20$

He now wants to know S_n of $1 + 3 + 6 + 10 + \ldots$, of which series he knows u_n to be $\dfrac{n(n+1)}{2}$.

This is the most difficult step of all his investigation.

But, knowing that

$$\text{if } u_n = 1, \text{ then } S_n = \frac{n}{1},$$

$$\text{if } u_n = \frac{n}{1}, \text{ then } S_n = \frac{n}{1} \cdot \frac{n+1}{2},$$

he might jump to the conclusion that

$$\text{if } u_n = \frac{n}{1} \cdot \frac{n+1}{2}; \qquad \text{then } S_n = \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3},$$

$$\text{if } u_n = \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3}, \text{ then } S_n = \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4};$$

at any rate these conclusions might be tested in particular cases, and when Septimus found his conclusions confirmed, he could prove them by Mathematical Induction.

He can confidently go on, making and summing new series, and knowing that with them he can find the number of sets of r things that can be chosen from n for all positive integral values of r and n.

He will now make out a number table :

1	2	3	4	5	6	7	u_n	S_n
1	1	1	1	1	1	1	1	n
1	2	3	4	5	6	7	n	$\frac{n}{1}\cdot\frac{n+1}{2}$
1	3	6	10	15	21	28	$\frac{n}{1}\cdot\frac{n+1}{2}$	$\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}$
1	4	10	20	35	56	84		
1	5	15	35	70	126	210		
1	6	21	56	126	252	462		
1	7	28	84	210	462	924		

and he will know how to write down :

(1) u_n and S_n of the series in any line ;

e.g. u_n of the 6th series will be $\frac{n(n+1)(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$,

S_n of the 6th series will be $\frac{n(n+1)\ldots(n+4)(n+5)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$.

(2) in how many ways a set of r things can be chosen from n, e.g. for sets of 6 things, the 7th row gives what is wanted ; the first term, 1, tells how many sets of 6 can be chosen from 6 ; the 2nd, 7, how many from 7 ; the 3rd, 28, how many from 8 ; and the order of the term is in each case 5 less than the total number of things from which the 6 are to be chosen.

Substituting $n - 5$ for n in the u_n of the sequence 1, 7, 28, i.e. in $\frac{n(n+1)(n+2)\ldots(n+5)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$, he then has that 6 things can be chosen from n in $\frac{(n-5)(n-4)(n-3)(n-2)(n-1)n}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \ldots}$ ways, and the form of this is more significant if written $\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cdot\frac{n-4}{5}\cdot\frac{n-5}{6}$.

Septimus has now achieved notable results in two branches of algebra : (1) the Summation of Series, (2) the Theory of Combinations, and a knowledge of the method of mathematical induction

would put beyond question the general formulæ he can now write down.

But his contemporaries were concerning themselves with another question, which at first sight shows no connexion with Septimus' investigations : how to write down in full the evaluation of $(x + a)^n$. If he took this up, he would probably try simple cases and find by multiplication that :

$$(1 + a)^2 = 1 + 2a + a^2$$
$$(1 + a)^3 = 1 + 3a + 3a^2 + a^3$$
$$(1 + a)^4 = 1 + 4a + 6a^2 + 4a^3 + a^4$$
$$(1 + a)^5 = 1 + 5a + 10a^2 + 10a^3 + 5a^4 + a^5.$$

But long work with his number-table has fixed the numbers and their sequence in his mind, and he recognizes that the coefficients of the different expansions

$$\begin{array}{ccccccccccc} & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

are the diagonals of his table.

He also sees that the coefficients of the next higher power are the sums of pairs of adjacent coefficients in the last power obtained, and the same addition law is fundamental in the formation of his table, so that the nth diagonal of the table, he can now be sure, gives the coefficients in the expansion of $(1 + a)^n$ where n is a positive integer. Moreover, he can, as we have seen, write down each of these coefficients in terms of n.

Ideas of negative and fractional indices have been coming into men's minds, and he might now try to expand $(1 + a)^{-n}$ where n is a positive integer.

By actual division he would find

$$\frac{1}{1 + a} = 1 - a + a^2 - a^3 + a^4 - \ldots$$

Dividing this result by $1 + a$ he would get

$$\frac{1}{(1 + a)^2} = 1 - 2a + 3a^2 - 4a^3 + 5a^4 - \ldots$$

and proceeding :

$$\frac{1}{(1 + a)^3} = 1 - 3a + 6a^2 - 10a^3 + 15a^4 - \ldots$$

and so on ; and he would see that the coefficients are numerically the rows of his number-table. He has now to prove that when any one of these expansions is divided by $1 + a$, each coefficient of the quotient is the sum of all the coefficients of the dividend which have come into the working. This he could do by dividing $(a - bx + cx^2 - dx^3 - \ldots)$ by $1 + x$ and getting the quotient $a - (a + b)x + (a + b + c)x^2 - \ldots$

Thus the law of development for the coefficients is the law of development for his number-table, and thus he will have established satisfactorily a formula for $(1 + a)^{-n}$ as well as for $(1 + a)^n$.

This investigation, which is given as a supposition of what might have been done, is indeed very much what actually was done in the seventeenth century. But whereas we have taken a few pages and a few minutes to sketch it, and have supposed it the work of one man, it was really the work of several great thinkers spread over a number of years that carried it from its first beginnings to its full development.

Some form of the number-table was made for expansions of $(1 + a)^n$ for certain numerical values of n in the middle of the sixteenth century.

Pascal employed the number-table partly for the expansion of $(1 + a)^n$ and partly for questions of probability, in 1653, and published his methods in 1665. The number-table is called, after him, **Pascal's Triangle**. By 1676 Newton had extended the expansion of $(1 + a)^n$, now known as the Binomial Theorem, to cases where n is a fraction.

We are not likely to find other examples so sustained and of such historical interest as the one we have just worked through, but whenever we do use the method of induction we shall not only be increasing the interest of the subject—we shall also be developing a taste and faculty for discovery.

CHAPTER XXII

THE PRINCIPLE OF PROPORTIONAL PARTS

THE tables ordinarily used in school work (those of square-roots, logarithms, etc.) can be compiled by the student. The labour of compiling a complete table would be too great to be worth while; but it is worth while to calculate a few values in different tables and to get some experience of the principles upon which the calculations are made and the tables compiled.

Logarithms can be evaluated by substituting values for x (where $-1 < x \leq 1$) in the series

$$\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots;$$

but the logarithms obtained are logs to base e (*see* p. 181). To reduce these to base 10 we use the relationship

$$\log_e 10 \times \log_{10} (1 + x) = \log_e (1 + x),$$

and since

$$\frac{1}{\log_e 10} \text{ (usually written } \mu) = \cdot 43429448 \ldots,$$

logarithms to base 10 and correct to 7 figures can be obtained from

$$\log_{10} (1 + x) = \cdot 43429448 \left\{ x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \right\};$$

x is chosen to meet two requirements: (1) It should be such as to make the evaluation of x^2, x^3, . . . as simple as possible, (2) it should be as small as possible, so as to make the series rapidly convergent.

Putting $x = -\frac{1}{10}$, we obtain log 9 and thus log 3.

Now putting $x = \frac{1}{80}$, we obtain log $\frac{81}{80}$; but log 9 having being found, we can obtain log 8, and thus log 2, and so on.

Sines and cosines of angles can be obtained from the series

$$\sin x = \theta - \frac{\theta^3}{1 . 2 . 3} + \frac{\theta^5}{1 . 2 . 3 . 4 . 5} \ldots,$$

$$\cos x = 1 - \frac{\theta^2}{1 . 2} + \frac{\theta^4}{1 . 2 . 3 . 4} - \ldots,$$

where x gives the measure in degrees and θ is the circular or radian measure. x and θ are connected by the relationship $\theta = \frac{\pi x}{180}$.

The above series enable us to calculate sines and cosines of

an angle to any degree of accuracy if we take a sufficiently accurate value of π.*

Now if log x and sin x were linear functions of x, i.e. if for the equal increases in the value of x there were equal increases in the value of the function, it would only be necessary to calculate two logs or two sines; all others could be obtained by proportion. But this is not so, e.g. elementary geometry gives us that

$$\sin 60^\circ = \frac{\sqrt{3}}{2} = \cdot 8660 \ldots$$

and

$$\sin 30^\circ = \frac{1}{2} = \cdot 5000 \ldots$$

If the sine graph were a straight line (representing a linear function), sin 45° would be $\frac{1}{2}$ ($\cdot 8660 + \cdot 5000$), i.e. $\cdot 6830$; it is actually $\frac{1}{\sqrt{2}}$, which is $\cdot 7071$.

It might appear to be necessary, then, to work out independently every fact that is tabulated in the tables. But this is not so, as we shall see.

In the ordinary 4-fig. log tables 9,000 facts are tabulated. This would not be possible on the double page of an ordinary book, unless (1) a very convenient form of tabulation were employed, and (2) the so-called difference columns decupled the numbers of facts on the rest of the pages. We shall consider the part played by the difference columns.

Taking log 3 to be $\cdot 4771213$ and putting x successively $\frac{1}{30}$, $\frac{3}{30}$, $\frac{5}{30}$, $\frac{6}{30}$, $\frac{9}{30}$ (i.e. $\cdot 3$) in the logarithmic series given above, we should get the logs of $3 \cdot 1$, $3 \cdot 3$, etc., which can be tabulated thus:

Nos.	Logs	Average Differences
30	4771	0143
31	4914	
32		0136
33	5185	
34		0128
35	5441	0122
36	5563	
37		0116
38		
39	5911	

* Log and sine tables were originally compiled by other means some generations before these series were known, and it must not be supposed that at the present day new tables, if required, would be compiled by the direct process of substitution; other methods are more expeditious.

The increase in the logarithm for unit increase in the number is not the same. The increases themselves diminish. This we should have expected from our knowledge of the graph of logarithms.

To 2-fig. but not to 3-fig. accuracy the differences are constant. The log graph for the range 30 to 39 can be regarded as a straight line to 2-fig. accuracy, but not to more.

Now again substituting $\frac{1}{300}$, $\frac{3}{300}$ $\frac{5}{300}$, $\frac{6}{300}$, $\frac{9}{300}$ for x in the series for $\log_{10}(1 + x)$, we obtain the logs of 3·01, 3·02, etc., and tabulate them :

Nos.	Logs	Average Differences
300	4771	0015
301	4786	
302		0014
303	4814	
304		$0014\frac{1}{2}$
305	4843	
306	4857	0014
307		
308		$0014\frac{2}{3}$
309	4901	

The differences are now constant to 3 figures and nearly so to 4.

For the range 300 to 309 the log graph may be regarded as a straight line to 3-fig. accuracy, and if it is regarded as a straight line to 4-fig. accuracy, the error will be very small. Let us consider this error.

The average difference over the range 300 to 309 is $0014\frac{4}{9}$; in only one case does a particular difference differ from this by more than $\frac{1}{2}$; if log 301 were obtained from log 300 by adding 0014, it would be 4785 instead of 4786. It would not be safe to obtain log 302, log 304, etc., by adding 0014 to log 301, log 303, etc. although the error would not exceed 0001.* Consequently, the logs of 302, 304, etc., are calculated independently, and a complete tabulation is made of logs of all 3-fig. numbers.

But in general it is safe to obtain logs of 4-fig. numbers by using average differences obtained from the average differences for 3-fig. numbers, i.e. we can regard the graph of logs from 300 to 301 as a straight line to 4-fig. accuracy.

The difference to add to log 3000 to give log 3001 should be 0001·43, to give log 3002 should be 0002·87, and so on, the liability

* In the earlier parts of the table, e.g. for logs of numbers from 100 to 200, it is even less safe, as the differences are greater, and themselves differ from the average difference by more than 0001.

to error being not greater than $0000\frac{1}{10}$, and this is less than the $0000\frac{1}{2}$,* which may always occur in 4-fig. approximation.

The 4th figure to be added to the logarithm for additions of 1, 2, 3, etc., in the number will be the corresponding multiples of 1·43. That is, for 1, 2, 3, 4, 5, 6, 7, 8, 9 in the 4th figure of a number within the range 3000 to 3010 we should add to the 4th place of the logarithm 1, 3, 4, 6, 7, 9, 10, 11, 13 ; and these, obtained by proportion, are the facts given in the difference columns. They can be applied over the whole range from 3000 to 3099.

Thus, instead of knowing merely the logarithms of the 3-digit numbers 300 to 309, we know the logarithms of the 4-digit numbers 3000 to 3099. Ten facts have become a hundred by the use of the difference columns.

In general, whenever for a range of values of x, the function of x exhibits the linear property accurately to any number of figures, to that number of figures we can find by proportion the value of the function of x for any value of x within the range.

If now we look up 5-fig. tables we have

Nos.	Logs	Diffs.	Nos.	Logs	Diffs.
300	47712		3000	47712	
		145			15
301	47857		3001	47727	
		144			14
302	48001		3002	47741	
		143			15
303	48144		3003	47756	
		143			14
304	48287		3004	47770	
		143			14
305	48430		3005	47784	
		142			15
306	48572		3006	47799	

i.e. the differences over the range 300–306 show a steady decrease which however would produce at the greatest an error of 1 in the 5th place in the logarithm if used to get logs of 4-digit numbers within that range. The differences over the range 3000–3006 show a very common vacillation about a mean difference.

Tabulation of 7-fig. logs (p. 277) for the same ranges of numbers will explain this vacillation, and will answer other questions that may have occurred to the reader.

Other tables could be similarly treated and investigated.

In some parts of some tables the difference columns cannot be satisfactorily compiled.

Take, for example, the first line of the 4-fig. log tables (p. 277). The use of the difference 43 to interpolate logs of numbers between 1080 and 1090 would lead to a possible error of 2 and in one case 3.

* As an extreme example of the liability to error in the last significant figure tabulated, take log 9·321 in 4-fig. tables—it is ·9694 ; but antilog ·9694 is given as 9·319.

Nos.	Logs	1st Diff.	2nd Diff.
300	4771213	14452	
301	4785665	14404	— 48
302	4800069	14357	— 47
303	4814426	14310	— 47
304	4828736	14262	— 48
305	4842998	14216	— 46
306	4857214		
3000	4771213	1447	
3001	2660	1447	
3002	4107	1446	
3003	5553	1446	
3004	6999	1446	
3005	8445	1445	
3006	9890		
30000	4771213	144	
30001	1357	145	
30002	1502	145	
30003	1647	145	
30004	1792	144	
30005	1936	145	
30006	2081		

Nos.	Logs	Diffs.
100	0000	43
101	0043	43
102	0086	42
103	0128	42
104	0170	42
105	0212	41
106	0253	41
107	0294	40
108	0334	40
109	0374	

To decrease this liability to error two sets of differences are often given—one for the range 1000 to 1040 and one for the range 1040 to 1090.

In some recent tables a more complete tabulation of these

logs is given; thus in MATHEMATICAL TABLES AND FORMULÆ by P. ABBOTT all the 4-fig. numbers from 1000 to 1999 are given to 5-fig. accuracy.

Again, in the table of tangents of angles, for angles near to 90° the difference columns are not filled up; the reader will see at once why. When the tangent of an angle such as 86° 40′ is needed, a fair value can be obtained from a graph of tangents of angles covering a range of ½° on each side of it, say 86° 12′, 86° 18′, 86° 24′, 86° 30′, 86° 42′, 86° 48′, 86° 54′, 87°, 87° 6′.

The principle on which the difference columns are compiled is called **The Principle of Proportional Parts or Differences**. We will exhibit it graphically.

Suppose that a function can be represented by the straight

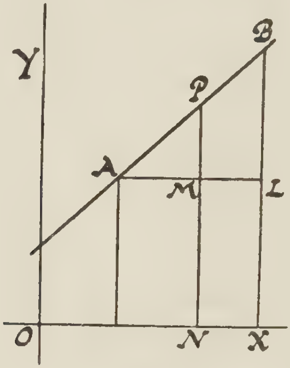


FIG. 242

line AB, and A and B are two determined points on it, representing known values, k_1 and k_2, of the function for known values, h_1 and h_2, of x. Let P (x, y) be any other point on it. Then if x is known y can be found. Let BL and PM be parallel to OY and AML parallel to OX.

Then by similar triangles, $\dfrac{\text{PM}}{\text{BL}} = \dfrac{\text{AM}}{\text{AL}}$,

i.e. $$\frac{y - k_1}{k_2 - k_1} = \frac{x - h_1}{h_2 - h_1},$$

and by hypothesis y is the only unknown quantity. Therefore y is obtained by solving a simple equation. The equation states that the ratio of the difference of values of y in two cases = the ratio of the difference of the corresponding values of x.

Now suppose that we are dealing with a function that is not

linear, but is continuous. Let A and B be two determined points on its graph (Fig. 243).

Let ON represent any value of x for which the corresponding value of the function is to be found. Let NPQ be the ordinate meeting the graph in P and the chord AB in Q.

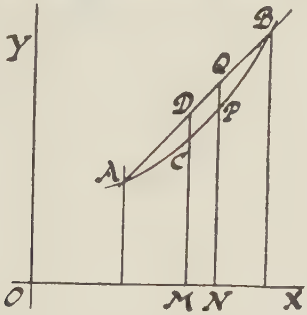


FIG. 243

Then by the principle of proportional parts the value of $f(x)$ would be obtained as QN, but it is PN. The error is QP.

Speaking somewhat vaguely, we might say that the nearer the arc APB is to being straight the less is the error QP.

To put it more mathematically:

If C is another point on the arc, DCM the corresponding ordinate, and if we have found that the theory of proportional differences holds for A, C, and B to any degree of approximation, then DC is negligible, and in these circumstances we can generally regard the error QP as being negligible also to that degree of approximation.

The principle can be applied to the approximate solution of equations; it gives a very rapid solution of cubics, biquadratics, and even of quadratics in cases where the formal method is cumbrous in application; it also solves equations for which there is no well-recognized method.

But here, instead of finding the value of a function for a known value of x, we are finding a value of x for a known value of the function. The principle of the procedure may be shown graphically thus:

Let ASPB (Fig. 244) represent the function, OR the given value. If RQP parallel to OX meets the graph in P, then RP is the value of x required.

Find by trial two approximate values of x, OM and ON, such that the corresponding values of the function may be respectively greater and less than OR. By proportional differences find the value of x corresponding to OR. This value of x will be represented by RQ, Q being the intersection of RP and the chord AB. The error in taking RQ as a solution is QP.

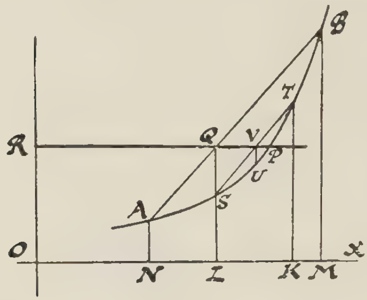


FIG. 244

Let the ordinate QL meet the curve in S. OL = RQ and SL is the corresponding value of the function. Take another neighbouring value of x, OK, so that RP is between OL and OK in magnitude, and let KT, the ordinate, meet the graph in T. Then by " proportional differences " applied to the values (OL, LS) and (OK, KT), a second approximation RV is obtained for x, V being the intersection of RP and the chord ST. In general, the new error VP, is much less than QP.

RV and another value of x being chosen as the next trial pair, and the method of " proportional differences " employed in the same way as above, a third approximation is obtained. The process can be repeated until a sufficiently accurate solution is obtained. When the solution is correct to the degree of accuracy required, three points on the graph corresponding to three neighbouring values of x will be linear to that degree of accuracy.

Examples will make this clearer. We shall use for the purpose three well known problems.

Example 1.

A ladder 20 *feet long leaning against the wall of a house just touches the top of a wall* 10 *feet high, distant* 3 *feet from the house. How high above the ground is the top of the ladder ?*

Let the height be $(10 + x)$ feet. Then by similar triangles and Pythagoras' Theorem,

$$(x + 10)^2\left(\frac{x^2 + 9}{x^2}\right) = 20^2.$$

Simplified, this equation becomes a biquadratic, which is cumbrous to solve.

The equation may be written

$$\frac{x + 10}{x}\sqrt{x^2 + 9} = 20\,;$$

i.e. we have to find a value of x for which the value of the function $(x + 10)\sqrt{1 + \frac{9}{x^2}}$ is 20.

There will be one solution where $3 < x < 10$, i.e. where the angle of slope is $> 45°$; and one where y (the distance of the foot of the ladder from the foot of the wall) > 10 but < 17, i.e. where the angle of slope $< 45°$.

Let us find the first solution—the other can be obtained in the same way. Try

$$f(7) = \frac{17}{7}\sqrt{58} = 2\tfrac{3}{7}\,(7\cdot616) = 18\cdot50,$$

$$f(8) = \frac{18}{8}\sqrt{73} = 2\tfrac{1}{4}\,(8\cdot544) = 19\cdot22,$$

$$f(9) = \frac{19}{9}\sqrt{90} = 2\tfrac{1}{9}\,(9\cdot487) = 20\cdot03;$$

$f(9)$ is too great, $f(8)$ too small.

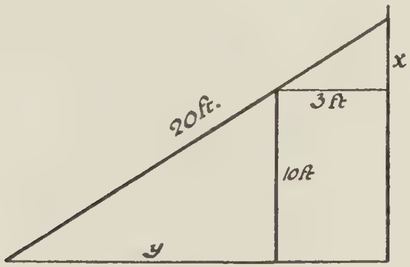


FIG. 245

Tabulate

$$\begin{aligned} f(8) &= 19\cdot22, \\ f(8+\alpha) &= 20, \\ f(9) &= 20\cdot03. \end{aligned}$$

By proportional parts,

$$\frac{(8+\alpha)-8}{9-8} = \frac{20-19\cdot22}{20\cdot03-19\cdot22},$$

i.e.

$$\begin{aligned} \alpha &= \frac{\cdot78}{\cdot81} \\ &= \cdot96. \end{aligned}$$

Try

$$f(8\cdot9)\ ;\ \text{it} = \frac{18\cdot9}{8\cdot9}\sqrt{88\cdot21} = 19\cdot95,$$

and is too small, as we should have expected.

Tabulate

$$\begin{aligned} f(8\cdot9) &= 19\cdot95 \\ f(8\cdot9+\beta) &= 20 \\ f(9) &= 20\cdot03. \end{aligned}$$

By proportional parts,

$$\frac{(8\cdot9+\beta)-8\cdot9}{9-8\cdot9}=\frac{\cdot05}{\cdot08},$$

i.e.

$$\frac{\beta}{\cdot1}=\frac{5}{8},$$

$$\beta=\cdot063.$$

Try

$$f(8\cdot96)\text{ ; it }=\frac{18\cdot96}{8\cdot96}\sqrt{89\cdot28}=20\cdot00,$$

and

$$f(8\cdot97)=\frac{18\cdot97}{8\cdot97}\sqrt{89\cdot46}=20\cdot00.$$

i.e. to 4-fig. accuracy the height is 18·96 or 18·97 feet.

So far mental arithmetic and 4-fig. tables have sufficed. To get a more accurate result we must use 5-, 6- or 7-fig. tables, and it will be convenient to use the equation in the form

$$\log(x+10)-\log x+\tfrac{1}{2}\log(x^2+9)=1\cdot3010300,$$

calling it

$$\varphi(x)=1\cdot3010300.$$

Now

$$\varphi(8\cdot96)=1\cdot3009113,$$
$$\varphi(8\cdot96+\gamma)=1\cdot3010300,$$
$$\varphi(8\cdot97)=1\cdot3010916,$$

whence

$$\frac{\gamma}{\cdot01}=\frac{1187}{1803}>\cdot6\text{ but }<\cdot7.$$

Now

$$\varphi(8\cdot966)=1\cdot3010195,$$
$$\varphi(8\cdot966+\delta)=1\cdot3010300,$$
$$\varphi(8\cdot967)=1\cdot3010373.$$

As the differences are 3-fig. differences, i.e. of the same order as the differences in the 7-fig. log tables, we will try

$$\varphi(8\cdot968)=1\cdot3010554,$$
$$\varphi(8\cdot968)-\varphi(8\cdot967)=\cdot0000181,$$
$$\varphi(8\cdot967)-\varphi(8\cdot966)=\cdot0000178.$$

Allowing for possible errors in the 7th decimal place, we see that to 7-fig. accuracy $\varphi(x)$ is approximately linear over the range $x = 8\cdot966$ to $8\cdot968$.

Now $\dfrac{\delta}{\cdot001}=\dfrac{105}{178}$ gives δ nearly ·0006 ;

and 8·9666 is therefore a good 5-fig. solution, and may be correct even farther ; φ (8·9666) is found to be 1·3010298, and we can say that x is approximately 8·966600 to 7 figures and $x + 10$ is 18·966600.

The computation has been omitted so that the procedure may be clearer. In the other examples we shall give the working more fully.

We may first note, however, that if we had taken θ to be the angle of slope of the ladder, we could have solved the equation,

$$10\operatorname{cosec}\theta+3\sec\theta=20\text{ ;}$$

and, θ being found, 20 sin θ gives the height required. This, in

point of fact, involves very much simpler computation, but as the remaining examples are trigonometrical equations, it seemed preferable to give here an example of the application of the principle to a troublesome algebraical one.

Example 2.

A goat is tethered to the fence surrounding a circular field of radius R. *How long must his tether be to enable him to graze half the field?*

Let O be the centre of the field, B the point where the tether rope is fixed, BP the length of the rope, and the circular arc PA the boundary of the goat's grazing.

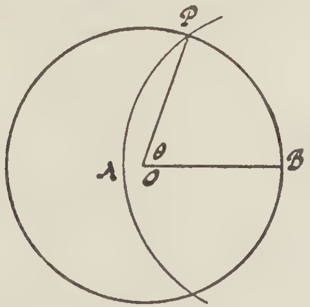


FIG. 246

Let POB be $\theta°$.

The rope's length is $2R \sin \frac{\theta}{2}$.

The area grazed over is

$$\pi R^2 \left\{1 - \frac{\sin\theta}{\pi} - \frac{180 - \theta}{180} \cos\theta\right\}, \text{ and this } = \tfrac{1}{2}\pi R^2.$$

We have to solve

$$\frac{\sin\theta}{\pi} + \frac{180 - \theta}{180} \cdot \cos\theta = \tfrac{1}{2},$$

i.e. $$\sin\theta + (\pi - \varphi)\cos\theta = \tfrac{1}{2}\pi \text{ , } \qquad \text{(I)}$$

where φ is the circular measure of θ

say $$f(\theta) = 1{\cdot}5708 \text{ (to 5 figures).}$$

Now, simple considerations show that $60° < \theta < 90°$, and also that as θ increases, the area increases and $f(\theta)$ decreases.

Take $\theta = 70°$ and $80°$ as the first trial pair and use 4-fig. tables.

θ	$\sin\theta$	$(\pi - \varphi)$	$\cos\theta$	$(\pi - \varphi)\cos\theta$	$f(\theta)$
70°	·9397	1·9199	·3420	·6566	1·5963

The nearness of 1·5963 to $\frac{\pi}{2}$ shows that we have had the luck to get close to the value of θ. Instead of proceeding with 80°, we will try 71°.

θ	$\sin\ \theta$	$(\pi - \varphi)$	$\cos\ \theta$	$(\pi - \varphi) \cos\ \theta$	$f\ (\theta)$
71°	·9455	1·9024	·3256	·6194	1·5649

$$\begin{aligned} \text{From} \qquad f(70°) &= 1{\cdot}5963, \\ f(70 + \alpha)° &= 1{\cdot}5708, \\ f(71°) &= 1{\cdot}5649, \\ \text{we have} \qquad \alpha &= \frac{255}{314} \\ &= {\cdot}8° \\ \text{and } \alpha &= \text{nearly } 48'. \end{aligned}$$

Now, over the range of 70°to 71°, sin θ and cos θ are, to 4-fig. accuracy, linear functions of θ, and $\pi - \varphi$ is a linear function. Therefore, we can rely on 70° 48′ as being right to within 1′, and hence we can get the length of the tether to 4-fig. accuracy.

To get a more accurate determination, we turn to 7-fig. tables and tabulate for 70° 48′, 70° 49′, 70° 50′. We have to solve

$$f(\theta) = 1{\cdot}5707963.$$

We shall tabulate in columns instead of rows in order to show all the computation conveniently arranged.

θ	70° 48′		70° 48′		70° 50′	
$\pi-\varphi$	1·9058995		1·9056086		1·9053177	
$\log (\pi-\varphi)$	·2801000		·2800337		·2799675	
$\log \cos \theta$	$\bar{1}$·5170198		$\bar{1}$·5166569		1·5162936	
$\log (\pi-\varphi) \cos \theta$	$\bar{1}$·7971198		$\bar{1}$·7966906		$\bar{1}$·7962611	
$(\pi-\varphi) \cos \theta$		·6267869		·6261676		·6255487
$\sin \theta$		·9443764		·9444720		·9445675
$f(\theta)$		1·5711633		1·5706396		1·5701162

The differences are ·0005237 and ·0005234, that is, to 7-fig. accuracy $f(\theta)$ is approximately linear over the range 70° 48′ to 70° 50′, and therefore the application of the method of proportional differences will give 7-fig. accuracy.

Instead of finding θ, it may be advisable to apply proportion to the values of $f(\theta)$ found above and the corresponding lengths of the tether, viz., $2\text{R} \sin \frac{\theta}{2}$, thus

θ	$f(\theta)$	$2\sin\frac{\theta}{2}$
70° 48′	1·5711633	1·1585624
true value	1·5707963	(1·1585624 + x)
70° 49′	1·5706396	1·1587995

Then by proportional differences

$$\frac{3670}{5237} = \frac{x}{\cdot 0002371},$$

$$\text{and } x = \cdot 0001662,$$

and the length of the tether is 1·1587286R, the 8th figure being unreliable.

Example 3.

A rectangular patch of ground 1 *mile by* $\frac{3}{4}$ *mile consists of a rectangle of grass* 1 *mile by* $\frac{1}{2}$ *mile, and a rectangle of gravel* 1 *mile by* $\frac{1}{4}$ *mile. A horseman who can do* 12 *m.p.h. on grass and* 8 *m.p.h. on gravel is to cross it from one corner to the diagonally opposite corner; find the shortest time in which he can do it.*

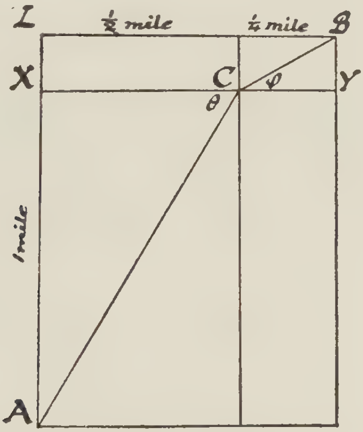


FIG. 247

Let C be the point where the horseman crosses the boundary between grass and gravel, and let XCY be parallel to the short side.

By taking the distance of C from LB to be x miles, we get that the time in hours

$$= \frac{1}{12}\sqrt{(1-x)^2 + \tfrac{1}{4}} + \frac{1}{8}\sqrt{x^2 + \tfrac{1}{16}}.$$

The minimum value of this can be found either by the methods of the differential calculus or (approximately) by graphical methods.

The method given here is used in order to show the application

of the principle of proportional parts to a problem involving two unknowns.

Let $\angle$ XCA be θ and BCY be φ.

If a ray of light were travelling through two different media along AC, CB such that its speeds along AC and CB were in the ratio of 3 to 2, the student of optics would know that the time along AC, CB would be a minimum if

$$\frac{\sin\theta}{3} = \frac{\sin\varphi}{2} \quad \ldots\ldots\ldots\ldots\ldots \quad \text{(I)}$$

This is the ordinary law for refracted light. We assume this and apply it to the analogous case of our example, asking the reader who is unacquainted with optics to accept it for the sake of the following method of solving a pair of simultaneous equations.

$$AL = CX \tan\theta + CY \tan\varphi.$$

Working with a quarter-mile as unit we have

$$2\tan\theta + \tan\varphi = 4 \quad \ldots\ldots\ldots\ldots \quad \text{(II)}$$

and the time taken is

$$\left(\frac{5}{2}\sec\theta + \frac{15}{8}\sec\varphi\right) \text{ minutes} \quad \ldots\ldots\ldots \quad \text{(III)}$$

We have to solve (I) and (II) for θ and φ and substitute in (III).

Now $\sin\theta > \dfrac{AL}{AB} > \cdot 8$ and is of course < 1.

We shall first tabulate for values of $\sin\theta$ that are simple multiples of 3, and using (I) we shall get values of $\sin\varphi$.

We shall thus have trial values of θ and φ to substitute in (II). Here clearly $\tan\theta < 4$.

Writing $f(\theta, \varphi)$ for $2\tan\theta + \tan\varphi$ we have

$\sin\theta$	$\sin\varphi$	θ	φ	$2\tan\theta$	$\tan\varphi$	$f(\theta, \varphi)$
·81	·54	54° 6′	32° 41′	2·7628	·6416	
·9	·6	64° 10′	36° 52′	4·1310 which is too big		
·84	·56	57° 8′	34° 3′	3·0956	·6758	3·7714
·87	·58	60° 27′	35° 27′	3·5278	·7120	4·2398

And by proportional parts $\sin\theta$ is nearly ·855.

$\sin\theta$	$\sin\varphi$	θ	φ	$2\tan\theta$	$\tan\varphi$	$f(\theta, \varphi)$
·855	·57	58° 46′	34° 45′	3·2982	·6937	3·9919
·8553	·5702	58° 48′	34° 46′	3·3024	·6941	3·9965
·8556	·5704	58° 50′	34° 47′	3·3066	·6945	4·0011

And the differences to 4-fig. accuracy are equal.

Tabulating for $f(\theta, \varphi)$ and the corresponding value of the time (i.e. $\frac{5}{2}\sec\theta + \frac{15}{8}\sec\varphi$) we have

$f(\theta, \varphi)$	$\frac{5}{2}\sec\theta$	$\frac{15}{8}\sec\varphi$	Time.
3·9919	4·8215	2·2819	7·1034
3·9965	4·8260	2·2824	7·1084
4·0000			$(7{\cdot}1084 + t)$
4·0011	4·8305	2·2830	7·1135

The value of the time corresponding to $f(\theta, \varphi) = 3{\cdot}9919$ is worked out to confirm that the time is also a linear function to 4-fig. accuracy over the range used; and t is determined from the last 3 rows by proportional differences between values of $f(\theta, \varphi)$ and of the time

$$\frac{t}{\cdot 0051} = \frac{35}{46},$$
$$t = \cdot 0038.$$

Therefore the shortest time is 7·1122 minutes, i.e. $7'\,6\frac{3}{4}''$, very nearly, or to stop-watch accuracy, $7'\,6\frac{4}{5}''$

This is not the place to describe ways of using the method to advantage in the case of different functions, nor to recount the advantages of using the method. The student's own practice will accumulate a useful experience in these respects. This one advantage, however, should be mentioned, that the method is self-checking and that a mistake at one stage is inevitably corrected as the work proceeds; this, by giving confidence, adds considerably to the speed of the method.

CHAPTER XXIII

PARADOXES AND FALLACIES

What is truth? said jesting Pilate and would not stay for an answer.—BACON.
There is no error so crooked but it hath in it some lines of truth.—TUPPER.

THE appeal of the paradox, like that of much that is humorous, is found in the shock of surprise; its mathematical value is to emphasize certain ideas or principles which may be elusive in application.

If, for example, the proof of a geometrical proposition depends on the particular position of points or lines obtained in the construction, the neglect to prove the position may result in something unexpected.

This can be shown in the well-known fallacy, *that a scalene triangle is equilateral.*

Let ABC be a scalene triangle, AO the bisector of $\angle$ CAB, DO the right bisector of the base, meeting AO in O. Draw OM, ON perpendicular to AC and AB. Join OB and OC.

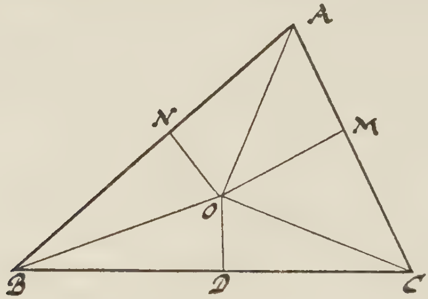


FIG. 248

Then $\triangle$s ANO and AMO are congruent,

$$\therefore \quad AN = AM \text{ and } NO = OM.$$

The $\triangle$s BOD and COD are congruent,

$$\therefore \quad BO = OC.$$

In the right-angled $\triangle$s BON and COM we have proved that BO = OC and NO = OM,

$$\therefore \quad BN = CM.$$

But we have also proved that AN = AM,

$$\therefore \quad \text{by addition } AB = AC;$$

and similarly we could prove that AB = BC.

If a number of carefully drawn figures were constructed it might appear that this result depends on the assumption that O is within the triangle. Let this be admitted, and let us use a figure in which O falls outside. We can apply to our new figure the same

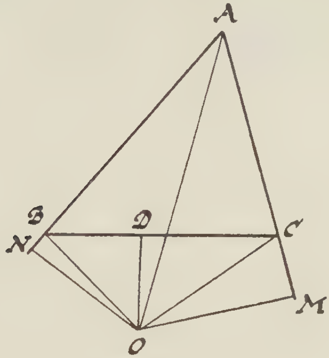


FIG. 249

proof as far as the last stage, i.e. we get

$$AN = AM,$$

and $BN = CM$,

then by subtraction $AB = AC$.

If O falls neither inside nor outside the triangle, but on BC, the first proof holds, provided that both ∠ ABC and ∠ ACB are acute.

But if one is obtuse, one of the ⊥s ON, OM is within and the other outside the triangle, and the last step of the proof fails.

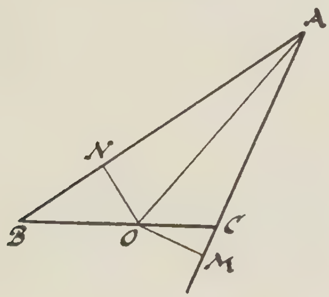


FIG. 250

This gives us the clue to the detection of the fallacy. We have assumed in our proof that O is in such a position that both

N and M are either (1) on the same side of BC as A, or (2) on the side remote. If one is on the same side and the other on the far side the proof fails; and it fails only if this is so. We require, then, to prove that the construction gives such a position.

Let the bisector of $\angle$ CAB meet the circum-circle in O. Then O is the mid-point of the arc BOC (the arc conjugate to BAC). But the right bisector of BC passes through the mid-point of this arc. Thus the position of O is exactly determined.

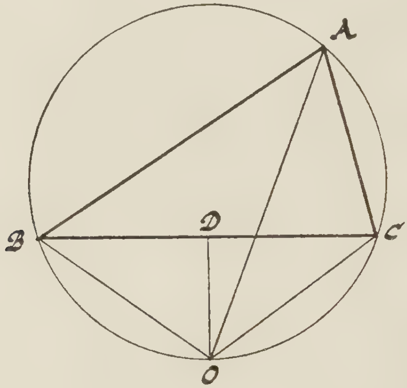


FIG. 251

Now ABOC is a cyclic quadrilateral, and since AB $\neq$ AC, AO is not a diameter of the circle. Therefore one of the angles ABO, OCA is acute and the other obtuse, and therefore the foot of one perpendicular is in the side and the foot of the other is in the side produced.

One or two of the following fallacies will appear to be trivial, but they are included as likely to provide a little amusement and to provoke clear thinking in the attempt to point out the exact step in the argument which is wrong. The others emphasize points important in mathematical theory.

The first two are the celebrated paradoxes of ZENO (of Elea, in Italy, 495–435 B.C.).

1. Achilles can run 10 times as fast as a tortoise; if he give the tortoise 1,000 yards start he can never catch it. For when he has run the 1,000 yards the tortoise is still 100 yards ahead; when he has run the 100 yards, the tortoise is still 10 yards ahead, and so on. Thus he can get nearer to the tortoise, but can never overtake it.

2. An arrow cannot move where it isn't, and it does not move where it is; therefore an arrow cannot move.

3. No horse has two tails. Every horse has one more tail than no horse; therefore every horse has 3 tails.

4. A bottle half empty = a bottle half full. But doubles of equals are equal, therefore a bottle quite empty = a bottle quite full.

5. If $a = b$,

$$a^2 = ab.$$

Subtracting b^2, $a^2 - b^2 = ab - b^2$.

Factorizing, $(a + b)(a - b) = a(a - b)$.

Dividing by $a - b$, $a + b = a$.

Putting $b = a$ $2a = a$;

$$\therefore 2 = 1.$$

6. $10 > 7$.

Subtracting 14, $-4 > -7$.

Squaring, $16 > 49$.

6a. As in 6, $-4 > -7$,

but -4 is a part of -7;

$\therefore$ the part is greater than the whole.

7. If we divide 1 by different numbers the quotient is greater as the number is less.

Now $-2 < 0 < 2$.

$$\therefore -\frac{1}{2} > \frac{1}{0} > \frac{1}{2},$$

i.e.

$$-\frac{1}{2} > \infty > \frac{1}{2},$$

8. $\frac{1}{2} > \frac{1}{4}$.

Taking logs, $\log \frac{1}{2} > 2 \log \frac{1}{2}$.

Dividing by $\log \frac{1}{2}$, $1 > 2$.

9.	1 mile	7 furlongs	39 poles	5 yds.	2 ft.	9 inches
Multiply by 2,	4 ,,	0 ,,	0 ,,	0 ,,	2 ,,	6 ,,
Divide by 2,	2 ,,	0 ,,	0 ,,	0 ,,	1 ,,	3 ,,

the less = the greater.

10. By long division

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

Putting $x = 2$, $-1 = 1 + 2 + 4 + 8 + \ldots$

Putting $x = -1$, $\frac{1}{2} = 1 - 1 + 1 - 1 + \ldots$

11. Let $S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$

Then $S = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + \ldots$

$$= \frac{1}{1 \times 2} + \frac{1}{3 \times 4} + \ldots$$

and is therefore positive and so $\neq 0$.

Now $2S = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \ldots$

Collecting terms which have the same denominator

$$2S = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots$$
$$= S,$$
$$\therefore \quad 2 = 1, \text{ since } S \neq 0.$$

12. Solve $\dfrac{1 - x}{x^2 - 3x - 4} = \dfrac{2}{4 - x} + \dfrac{1}{x - 1}$.

Multiplying through by the common denominator, $(x^2 - 3x - 4)(4 - x)(x - 1)$,

$$x^3 - 6x^2 + 9x - 4 = x^3 - x^2 - 10x - 8,$$
$$5x^2 - 19x - 4 = 0,$$
$$(5x + 1)(x - 4) = 0,$$
$$x = -\tfrac{1}{5}$$
$$\text{or } x = 4$$

Test each solution by substituting in the equation given.

12a. Solve $\dfrac{7}{x^2 - x - 12} + \dfrac{1}{x^2 + 5x + 6} = \dfrac{6}{x^2 - 2x - 8}$.

Multiplying up by the common denominator $(x + 2)(x + 3)(x - 4)$,

$$7x + 14 + x - 4 = 6x + 18,$$
$$2x = 8,$$
$$x = 4.$$

Test this.

13. Solve

$$3x^2 - xy - y^2 = 139 \ldots\ldots\ldots\ldots (1)$$
$$x^2 + 2xy + 7y^2 = 70 \ldots\ldots\ldots\ldots (2)$$

Eliminating the numerical term

$$71x^2 - 348xy - 1043y^2 = 0,$$
$$(71x + 149y)(x - 7y) = 0,$$
$$x = -\frac{149}{71}y \text{ or } 7y.$$

Taking first $x = 7y$ and substituting in (2)

$$49y^2 + 14y^2 + 7y^2 = 70,$$
$$70y^2 = 7,$$
$$y = \pm 1.$$

Substituting $y = 1$ in (1)

$$3x^2 - x - 140 = 0,$$
$$(3x + 20)(x - 7) = 0,$$
$$x = 7 \text{ or } -6\tfrac{2}{3}.$$

Substitute in turn in (2), $x = 7, y = 1$ or $x = -6\frac{2}{3}$, $y = 1$. The second of these solutions does not satisfy.

14. Solve $\dfrac{\sin\theta}{1 + \cos\theta} = 2 - \cot\theta$.

That is $\dfrac{\sin\theta}{1 + \cos\theta} = 2 - \dfrac{\cos\theta}{\sin\theta}$

$$\therefore \quad \sin^2\theta = (1 + \cos\theta)(2\sin\theta - \cos\theta),$$
$$= 2\sin\theta + 2\sin\theta\cos\theta - \cos\theta - \cos^2\theta;$$

i.e.
$$1 + \cos\theta - 2\sin\theta - 2\sin\theta\cos\theta = 0,$$
$$\therefore \quad (1 + \cos\theta)(1 - 2\sin\theta) = 0,$$
$$\therefore \quad \sin\theta = \tfrac{1}{2},$$
$$\text{or } \cos\theta = -1.$$

Test both solutions by substituting in the original equation.

15.
$$-1 = -1;$$
$$\frac{1}{-1} = \frac{-1}{1}.$$

Taking the square root,
$$\frac{\sqrt{1}}{\sqrt{-1}} = \frac{\sqrt{-1}}{\sqrt{1}}.$$

By cross multiplication, $1 = -1$.

16. To prove that every obtuse angle is a right angle.

Let ABC be the obtuse angle.

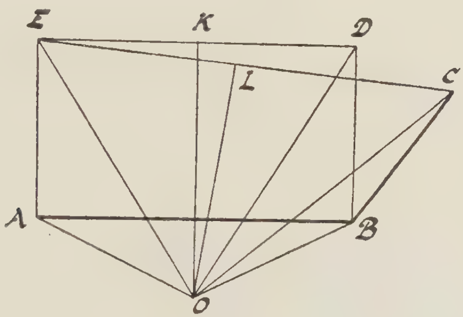


FIG. 252

Make ABD a right angle and cut off BD and BC equal to each other. Complete the rectangle ABDE.

Let the right bisectors KO and LO of ED and EC meet in O. Join OA, OE, OC, OB.

Now LO is the right bisector of EC, $\therefore$ EO = OC;

and KO is the right bisector of AB, $\therefore$ AO = OB.

In $\triangle$s EAO and COB,

$$EO = OC,$$
$$AO = OB,$$
and
$$EA = BC;$$
$$\therefore \quad \angle OAE = \angle OBC.$$

But $\qquad \angle OAB = \angle OBA.$

Subtracting $\qquad \angle EAB = \angle ABC,$

$\therefore$ the obtuse $\angle$ ABC is a right angle.

17. To prove that there is no "ambiguous case."

Let the $\triangle$s ABC and DEF have AB = DE, BC = EF and

$\angle$ BAC = $\angle$ EDF. We are accustomed to say that the triangles are not necessarily congruent with these data.

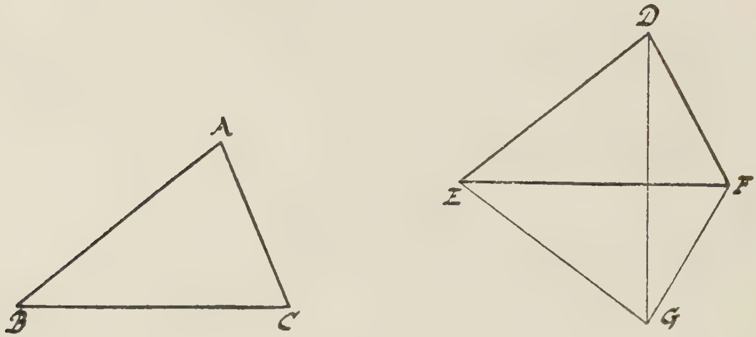


FIG. 253

On EF on the side remote from D construct $\triangle$ EGF congruent with ABC. Join DG.

$$
\begin{aligned}
\text{Now} \qquad & \text{ED} = \text{AB} \\
& \phantom{\text{ED}} = \text{EG}; \\
\therefore \qquad & \angle \text{EDG} = \angle \text{EGD}. \\
\text{But} \qquad & \angle \text{EGF} = \angle \text{BAC} \\
& \phantom{\angle \text{EGF}} = \angle \text{EDF}; \\
\therefore \quad \text{remaining } & \angle \text{GDF} = \text{remaining } \angle \text{DGF}; \\
\therefore \qquad & \text{DF} = \text{GF};
\end{aligned}
$$

therefore $\triangle$s EDF and EGF have three sides of one equal to three sides of the other and are congruent.

But $\triangle$ EGF was made congruent with $\triangle$ ABC,

$\therefore$ $\triangle$ EDF is congruent with $\triangle$ ABC,

i.e. there is no ambiguous case.

18. Take a piece of paper 8 cm. square. Its area is 64 sq. cm. Divide it into 4 parts as shown in Fig. 254 (1) and rearrange them as in Fig. 254 (2).

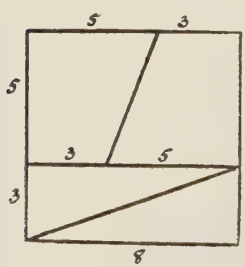


FIG. 254 (1)

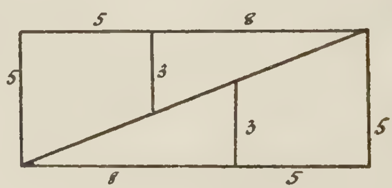


FIG. 254 (2)

The area is now 13×5, i.e. 65 sq. cm.;

$$\therefore \quad 64 = 65.$$

19. $\tan(90° + x) = -\tan(90° - x)$.

Put $x = 0$,

$$\tan 90° = -\tan 90°$$

20. Let ABC be an isosceles right-angled triangle having AB = BC. Let D, F, E be the mid-points of the sides.

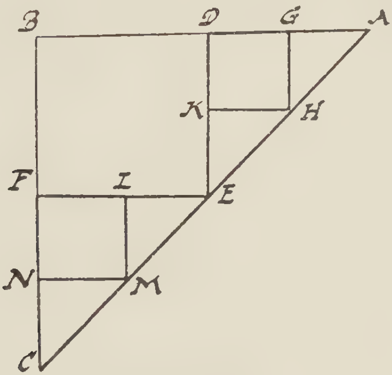


FIG. 255

Then ADE and EFC are similar isosceles right-angled triangles and AB + BC = AD + DE + EF + FC.

Make the same construction for the $\triangle$s ADE and EFC.

Then AB + BC = (AD + DE) + (EF + FC)
= AG + GH + HK + KE + EL + LM + MN + NC,

and by continuing the process indefinitely

AB + BC = AC.

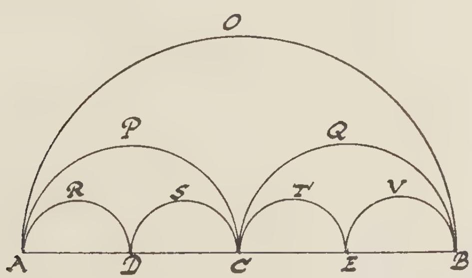


FIG. 256

By a similar method it can be shown that the diameter of a circle = the semi-circumference.

Draw semicircle AOB. Obtain C and D and E by bisections, and draw semicircles as in Figure. 256.

Then AOB = APC + CQB

= ARD + DSC + CTE + EVB, and so on.

= diameter AB.

21. Let AOB be an angle and AB and CD two arcs whose common centre is O.

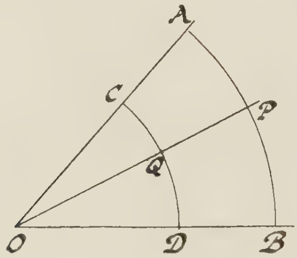


FIG. 257

Every line through O, such as OQP, meets each arc in one point. Therefore there are as many points in one arc as in the other, therefore the arcs are of the same length.

22. The centre of gravity of the uniform triangle ABC is at G in the median AD, such that GD = $\frac{1}{3}$AD.

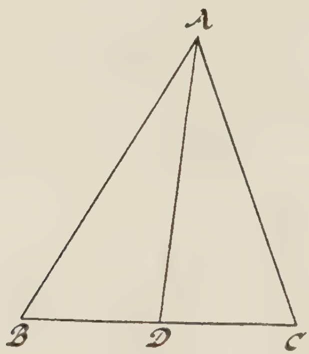


FIG. 258

Consider a triangle in which B and C coincide with D. Then the triangle becomes the straight line AD and its centre of gravity is the middle point, which is not G.

23. When a wheel rolls along a level plane, remaining in the same vertical plane, then for each revolution its centre is displaced horizontally a distance equal to the circumference.

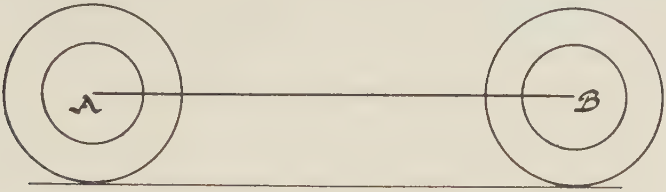


FIG. 259

Consider two wheels of different sizes with a common axle A. Let A move to B while the larger wheel rolls once along a plane. Then AB = circumference of the large wheel. But the smaller wheel has also made one revolution. Therefore AB = circumference of the smaller wheel. Therefore the wheels are equal.

24. To solve $x^3 - 3x^2 + 3x = 7$.

Differentiating $\quad 3x^2 - 6x + 3 = 0,$

$x = 1$ is the solution.

Check by substitution.

25. To every action there is an equal and contrary reaction (Newton's First Law of Motion).

Consider the case of a horse pulling a cart; if the horse pull the cart with a force P, the cart will exert an equal and opposite pull P.

These two equal and opposite forces maintain equilibrium. Therefore there can be no motion.

26.

$$\begin{aligned}
\int 2 \sin x \cos x \, dx &= \int 2 \sin x \, (d \sin x) \\
&= \sin^2 x. \\
\int 2 \sin x \cos x \, dx &= \int \sin 2x \, dx \\
&= -\tfrac{1}{2} \cos 2x\,; \\
\therefore\ \tfrac{1}{2} \cos 2x &= -\sin^2 x\,; \\
\text{but } \tfrac{1}{2} \cos 2x &= \tfrac{1}{2} - \sin^2 x\,; \\
\therefore\ \tfrac{1}{2} &= 0
\end{aligned}$$

27.

$$(-1)^2 = 1.$$

Taking logs, $\quad 2 \log(-1) = 0,$

$$\therefore\ \log(-1) = 0.$$

Nos. 26 and 27 are inserted although they are outside the range of work assumed in this book.

The detection of these fallacies is left to the reader; the interest of doing so will probably not end with discovering where

and what the flaw is ; it may lead to an investigation of precise details, e.g. in No. 18 it is fairly obvious that along the diagonal of Fig. 254 (2) there will not be a perfect fit, there will be a space whose area is 1 sq. cm. That there will be a space can be simply proved by similar triangles and the fallacy will be exposed ; but it may be interesting to go farther—to determine the shape of the gap and the measurements of its sides and angles and from these its area.

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